JET-LIKE QED PROCESSES :

ON GENERAL PROPERTIES OF IMPACT FACTORS

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ABSTRACT

In this article, general properties of impact factors involved in jet-like QED processes are explored. In particular, a general link is established between their helicity properties and their order of magnitude as defined by jet-like kinematics. Exact results are given for some processes in the strict forward direction and a general method is proposed to track orders in the calculation of multi-bremsstrahlung processes.

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1 Preliminaries

Let us first remind the way one defines an helicity frame for each particle taking part in a given process (see also Ref [**1**]).

The 4-vectors of the basis defining the lab frame will be denoted by $T,\ X,\ Y,\ Z$. In this basis, the 4-momentum p of a particle of mass m is developed as

$$p = E\ T + P\cos\theta\ Z + P\sin\theta\,(\cos\varphi\ X + \sin\varphi\ Y) \tag{1}$$

where $P = \sqrt{E^2 - m^2}$. We first introduce an helicity triad $X',\ Y',\ Z'$ of space-like 4-vectors associated with T, which achieves an helicity coupling scheme between the two time-like 4-vectors T and p :

$$\begin{gathered} X' = -\sin\theta\ Z + \cos\theta\,(\cos\varphi\ X + \sin\varphi\ Y)\,, \quad Y' = -\sin\varphi\ X + \cos\varphi\ Y \\ Z' = \cos\theta\ Z + \sin\theta\,(\cos\varphi\ X + \sin\varphi\ Y) \end{gathered} \tag{2}$$

The helicity frame associated with the 4-momentum p is then defined by the following 4-vectors

$$t = \frac{p}{m}, \quad x = X', \quad y = Y', \quad z = \sinh\chi\ T + \cosh\chi\ Z' \tag{3}$$

where $\cosh\chi = E/m$.

The Lorentz transformation $\mathcal{S}$ that tranforms the reference basis $T,\ X,\ Y,\ Z$ into the helicity basis $t,\ x,\ y,\ z$ is the product of three transformations : a boost along the $Z-$axis with rapidity χ, followed by a rotation of angle θ around the $Y-$axis, followed by a rotation of angle φ around the $Z-$axis.

Dirac spinors associated with the reference frame may be defined as follows. In the spinorial representation, spin operators associated with the reference basis $T,\ X,\ Y,\ Z$ are

$$S_X = \frac{1}{2}\gamma_5\gamma(X)\gamma(T), \quad S_Y = \frac{1}{2}\gamma_5\gamma(Y)\gamma(T), \quad S_Z = \frac{1}{2}\gamma_5\gamma(Z)\gamma(T) \tag{4}$$

where the notation $\gamma(V) = V_\mu\gamma^\mu$ is used. The reference Dirac spinors will be denoted by U_0^λ ($\lambda = \pm 1/2$). They satisfy the relations

$$\gamma(T)U_0^\lambda = U_0^\lambda, \quad S_z U_0^\lambda = \lambda U_0^\lambda, \quad \bar{U}_0^\lambda U_0^{\lambda'} = 2\,\delta_{\lambda'\lambda} \tag{5}$$

The Dirac spinors U^λ associated with the helicity basis $t,\ x,\ y,\ z$ are related to the reference Dirac spinors through the Lorentz transformation

$$\mathcal{S} = \mathcal{R}_Z(\varphi)\mathcal{R}_Y(\theta)\mathcal{H}_Z(\chi) \tag{6}$$

where, in spinorial representation,

$$
\begin{aligned}
\mathcal{R}_Z(\varphi) &= \cos(\frac{\varphi}{2}) - 2i\sin(\frac{\varphi}{2})S_Z \\
\mathcal{R}_Y(\theta) &= \cos(\frac{\theta}{2}) - 2i\sin(\frac{\theta}{2})S_Y \\
\mathcal{H}_Z(\chi) &= \cosh(\frac{\chi}{2}) + 2\sinh(\frac{\chi}{2})\gamma_5 S_Z
\end{aligned}
\tag{7}
$$

Explicitly, we have (using the notation $\uparrow$ for $\lambda = +1/2$ and $\downarrow$ for $\lambda = +1/2$, defining spinors $V = \gamma_5 U$ and here choosing the normalisation $\bar{U}^{\lambda} U^{\lambda'} = 2m\,\delta_{\lambda'\lambda}$)

$$
\begin{aligned}
U^{\uparrow} &= \cosh(\frac{\chi}{2})U'^{\uparrow} + \sinh(\frac{\chi}{2})V'^{\uparrow} \\
\text{where}\quad U'^{\uparrow} &= \sqrt{m}\left[\exp(-i\frac{\varphi}{2})\cos(\frac{\theta}{2})U_0^{\uparrow} + \exp(i\frac{\varphi}{2})\sin(\frac{\theta}{2})U_0^{\downarrow}\right] \\
U^{\downarrow} &= \cosh(\frac{\chi}{2})U'^{\downarrow} - \sinh(\frac{\chi}{2})V'^{\downarrow} \\
\text{where}\quad U'^{\downarrow} &= \sqrt{m}\left[\exp(i\frac{\varphi}{2})\cos(\frac{\theta}{2})U_0^{\downarrow} - \exp(-i\frac{\varphi}{2})\sin(\frac{\theta}{2})U_0^{\uparrow}\right]
\end{aligned}
\tag{8}
$$

If the considered particle is a lepton, its spinors will be defined as above. If that particle is a real photon, its 4-momentum k is no more $\propto t$ (the latter 4-vector being time-like), but should be considered as $k = E(t+z)$ (which is light-like), and its circular polarizations will be defined as

$$
\epsilon^{(\pm)} = \mp\frac{1}{\sqrt{2}}\,(x \pm iy) \tag{9}
$$

As regards the latter formula, let us notice the useful identity

$$
x = \cos\varphi\ X + \sin\varphi\ Y - \tan(\frac{\theta}{2})\,(Z + Z') \tag{10}
$$

that allows us to rewrite the circular polarizations in the form

$$
\epsilon^{(\pm)} = E^{(\pm)}\exp(\mp i\varphi) \pm \frac{1}{\sqrt{2}}\tan(\frac{\theta}{2})\,(\exp(-\chi)(t+z) - (T - Z)) \tag{11}
$$

using the relation $T + Z' = \exp(-\chi)(t+z)$.

If we make use of the gauge invariance of QED amplitudes, we see that we can drop the term $\propto t + z \propto k$ in the above expression, and redefine the circular polarizations of a real photon by

$$
\epsilon^{(\pm)} = \exp(\mp i\varphi)\left(E^{(\pm)} + \xi^{\star}(T - Z)\right),\ \text{where}\ \ \xi^{\star} = \mp\frac{1}{\sqrt{2}}\tan(\frac{\theta}{2})\exp(\pm i\varphi) \tag{12}
$$

It is easy to check that the latter polarization 4-vectors are still orthogonal to k. They are also orthogonal to $T - Z$.

2 On helicity properties of impact factors[2]

To be specific, let us consider the impact factor corresponding to the QED subprocess lepton + $\gamma^\star \to$ lepton + jet. This impact factor is generally written as a sum of terms like $\bar{U}_3 \mathcal{T} U_1$, where U_1 is the bi-spinor of the incoming lepton, U_3 is the bi-spinor of a lepton of the same species pertaining to the final jet ; $\mathcal{T}$ is a 4X4 transition matrix which is a succession of products of lepton propagator and γ-matrices describing vertices (the other terms completing the impact factor correspond to possible exchanges between final particles). We will denote by λ the helicity of the incoming lepton, by λ' that of the outgoing lepton and by Λ the total helicity of the remaining particles in the final jet.

Let us perform a rotation of angle Φ around the Z-axis of the lab frame. This amounts to a redefinition of X and Y transverse axes. Under such a transformation, wave functions undergo the change

$$\Psi_{\lambda_\psi} \to \exp(-i\lambda_\psi \Phi)\ \Psi'_{\lambda_\psi} \tag{13}$$

where Ψ'_{λ_ψ} is the transformed wave function, and we must have

$$J = \exp(i(\Lambda + \lambda' - \lambda)\Phi)\ J' \tag{14}$$

where J' is obtained from J by a simple redefinition of X and Y axes.

Since denominators of propagators entering into J possibly depend on azimutal angles only through relative combinations of those angles, they remain unchanged. These denominators will be left apart from the discussion regarding successive orders of approximation in the jet-like configuration. So, in the following, they should be considered as if they were simple constant coefficients.

Then, we may make the (very mild) assumption that J is an analytical function of transverse components p_{qx} and p_{qy} of 4-momenta, and also an analytical function of the lepton mass. Instead of the above "linear" transverse components, we may rather consider the "circular" combinations

$$z_q = p_{qx} + ip_{qy} \quad \text{and} \quad z_q^\star = p_{qx} - ip_{qy} \tag{15}$$

Such an assumption could be verified at least in the framework of jet-like kinematics where it could be possible to make the expansion

[2] The definition of an impact factor is given in refs [2], [3].

$$
\begin{aligned}
J\left(\{z_q\},\left\{z_q^\star\right\},m\right) \approx\ & J|_0 + m\ \frac{\partial J}{\partial m}|_0 + m^2\ \frac{\partial^2 J}{(\partial m)^2}|_0 + ... + \sum_q z_q\ \frac{\partial J}{\partial z_q}|_0 \\
& + \sum_q z_q^\star\ \frac{\partial J}{\partial z_q^\star}|_0 + \frac{1}{2}\sum_q z_q^2\ \frac{\partial^2 J}{\partial z_q^2}|_0 + \frac{1}{2}\sum_q z_q^{\star 2}\ \frac{\partial^2 J}{\partial z_q^{\star 2}}|_0 + \sum_{r<s} z_r z_s\ \frac{\partial^2 J}{\partial z_r \partial z_s}|_0 \\
& + \sum_{r<s} z_r^\star z_s^\star\ \frac{\partial^2 J}{\partial z_r^\star \partial z_s^\star}|_0 + \sum_{r<s} z_r z_s^\star\ \frac{\partial^2 J}{\partial z_r \partial z_s^\star}|_0 + \sum_{r<s} z_r^\star z_s\ \frac{\partial^2 J}{\partial z_r^\star \partial z_s}|_0 \\
& + m\ \sum_q z_q\ \frac{\partial^2 J}{\partial m \partial z_q}|_0 + m\ \sum_q z_q^\star\ \frac{\partial^2 J}{\partial m \partial z_q^\star}|_0 + \text{higher order terms}
\end{aligned} \tag{16}
$$

Here, the symbol $|_0$ means that the corresponding quantity is taken at $m = 0$ and all z_q and $z_q^\star$ equal to zero (except in denominators, as said above). Of course, the leading order from which the expansion (16) should start essentially depends on the subprocess considered.

We may perform the same kind of expansion in both sides of eq (2). Taking into account the fact that

$$z_q^\prime = \exp(i\Phi)\ z_q \quad \text{and} \quad z_q^{\prime\star} = \exp(-i\Phi)\ z_q^\star \tag{17}$$

we easily find, by simple identification of various terms, that we should have the constraints

$$\Lambda + \lambda' - \lambda = n - p \tag{18}$$

on coefficients of the expansion corresponding to p (for positive) factors z_r and n (for negative) factors $z_r^\star$. Such constraints automatically imply strong correlations between the relative values of helicities and the order of approximation. Let us consider some consequences.

1) For amplitudes without lepton-helicity-flip, we have $\lambda' = \lambda$, and thus $\Lambda = n - p$. For terms coming from the lepton mass only, we have $n = p = 0$ and, therefore, $\Lambda = 0$. As seen in appendix 6.2, such amplitudes are even functions of m, so that terms with odd exponents of m should be absent from the corresponding expansion.

We may then conclude the following.

- For the elastic vertex $\ell + \gamma^\star \to \ell'$ where the final jet is made up of the final lepton only, the leading contribution is $J|_0$, and mass correction arise only at second order. In addition, since $n = p$, corrections from z's and $z^\star$'s arise also at second order.

- Regarding the process $\ell + \gamma^\star \to \ell' + \gamma$.

  Now, we have $\Lambda_\gamma = \pm 1 = n - p$, or $n = p \pm 1$. This means that the leading order term cannot be $J|_0$ which is zero. The leading terms correspond to the cases

$n = 1,\ p = 0,\ \Lambda = +1$ (terms $\propto p_x - ip_y$) and $n = 0,\ p = 1,\ \Lambda = -1$ (terms $\propto p_x + ip_y$). Notice that corrections *from mass terms only* should cancel. The first mass corrections are of third order through terms involving $\partial^3 J/((\partial m)^2 \partial z)$ $(n = 1,\ p = 0,\ \Lambda = 1)$ or $\partial^3 J/((\partial m)^2 \partial z^\star)$ $(n = 0,\ p = 1,\ \Lambda = -1)$.

- Regarding the process $\ell + \gamma^\star \to \ell' + \gamma_1 + \gamma_2$.

  We here have $\lambda_1 + \lambda_2 = n - p$.
  For $\lambda_1 = \lambda_2 = +1$, then $n = p + 2$ and the leading order is given by terms $\propto z_r^\star\ z_s^\star$ $(n = 2,\ p = 0)$. In that case, mass corrections cannot arise without p_T corrections : the first corrections are of 4th order $(\sim m^2\ z_r^\star\ z_s^\star)$.
  For $\lambda_1 = \lambda_2 = -1$, we have $p = n + 2$ and leading contributions are $\propto z_r\ z_s$.
  For $\lambda_1 = -\lambda_2$, we have $n = p$. In that case can we have a leading term $J|_0$? The answer seems to be : no. This may be related to the fact that

  $$J \propto q_T \quad \text{as} \quad q_T \to 0 \tag{19}$$

  as can be derived from the current conservation relation $q^\mu J_\mu = 0$. We thus conclude that the leading terms are either terms $\propto m^2$ or terms $\propto z_r\ z_s^\star$.

2) Let us now turn to the case of lepton-helicity-flip amplitudes and take, for definiteness, $\lambda = -\lambda' = 1/2$. Then $\Lambda = n - p + 1$. In addition, we know that such amplitudes are odd functions of m and their corresponding expansions should not contain terms with even exponents of m.

- For the elastic vertex, we get $n = p - 1$ and the leading term is of second order $\propto m\ z$ $(n = 0,\ p = 1)$.

- For $\ell + \gamma^\star \to \ell' + \gamma$, we have $\Lambda_\gamma = \pm 1 = n - p + 1$ or $p = n + 1 \mp 1$ and a leading term $\propto m$ is obtained for $\Lambda_\gamma = 1$ when $n = p = 0$, with next corrections of 3rd order $(\sim m^3,\ \sim m\ z_r\ z_s^\star)$. For $\Lambda_\gamma = -1$, $p = n + 2$ and the leading terms in this case are of 3rd order $(\sim m\ z_r\ z_s)$. We see here the strong correlation between the helicity of the outgoing photon and that of the incoming lepton.

- For the process $\ell + \gamma^\star \to \ell' + \gamma_1 + \gamma_2$, we have $\lambda_1 + \lambda_2 = n - p + 1$.
  If $\lambda_1 = \lambda_2 = +1$, $n = p + 1$ and the leading terms correspond to $n = 1,\ p = 0$ and are $\propto m\ z^\star$ (2nd order).
  If $\lambda_1 = -\lambda_2 = +1$, $p = n + 1$, and the leading terms correspond to $n = 0,\ p = 1$ and are $\propto m\ z$ (2nd order).
  Finally, if $\lambda_1 = \lambda_2 = -1$, $p = n + 3$, and this time, the leading terms are of order 4 and are $\propto m\ z_q\ z_r\ z_s$ $(n = 0,\ p = 3)$. In this case we can say that to 4th order the corresponding amplitude is zero.

  Here again, we see a strong correlation between the helicity of an outgoing photon and that of the incoming lepton, for lepton-helicity-flip amplitudes.

So, it appears that most of helicity properties observed in specific calculations ([**2-3**]) can be simply explained in such a formalism. For example, the maximum change in helicity implies a maximum value of $|n-p|$, and consequently leading terms of the expansion should be of higher order.

Anyway, for a former discussion about jet-like kinematics, we suggest to speak immediately about "leading order calculations", instead of saying that in such calculations only terms of first order in m/E or $\theta' s$ are kept while neglecting higher orders, which is not the case for the process $\ell + \gamma^\star \to \ell' + \gamma_1 + \gamma_2$ where amplitudes are of 2nd order. This would be more cautious and maybe more clear.

3 On analyticity of impact factors as regards angular variables

Let us notice that in jet-like kinematics, numerators of transition amplitudes are expanded in terms of small quantities mass m and polar angles θ. Thus, instead of using the variables (15) we will use the new ones

$$z_k = \theta_k \exp\left(i\varphi_k\right) \;, \quad \text{and} \quad z_k^\star \tag{20}$$

to express impact factors. If we extract from the transition amplitudes the phase factors $\exp(i\lambda_{\rm out}\varphi_{\rm out})$ for outgoing particles and $\exp(-i\lambda_{\rm in}\varphi_{\rm in})$ for ingoing particles, we assert that the remaining factor is, as regards angular variables, a function of the $z's$ and of the $z^\star$'s only. This can be understood in the following way.

Spinors of the fundamental representation of spin 1/2 are written in the form

$$\begin{aligned} u^\uparrow &= \frac{\exp(-i\varphi/2)}{\sqrt{2}} \begin{pmatrix} \cos(\theta/2) \\ \exp(i\varphi)\sin(\theta/2) \end{pmatrix} \\ u^\downarrow &= \frac{\exp(i\varphi/2)}{\sqrt{2}} \begin{pmatrix} -\exp(-i\varphi)\sin(\theta/2) \\ \cos(\theta/2) \end{pmatrix} \end{aligned} \tag{21}$$

and spinors of higher spin may be constructed from tensorial products of the above spinors. They are thus function of $\cos(\theta/2)$ and $\sin(\theta/2)$. More precisely, the latter functions are factorized with azimutal phase factors $\exp(i\varphi/2)$ and $\exp(-i\varphi/2)$. For spin up spinor we may extract the phase factor $\exp(-i\varphi/2)$ and the remaining is a linear (spinor) combination of $\cos(\theta/2)$ and $\sin(\theta/2)\;\exp(i\varphi)$. But the expansions

$$\cos(\theta/2) = 1 - \theta^2/8 + ... = 1 - zz^\star/8 + ... \tag{22}$$

and

$$\sin(\theta/2)\;\exp(i\varphi) = \;\exp(i\varphi)\left(\theta/2 - \theta^3/48 + ...\right) = z/2 - z^2 z^\star/48 + ... \tag{23}$$

clearly show that they are analytic functions of the variables z and $z^\star$ as defined by (20). An analogous result holds for the spin down spinor. As for photons, we have (applying gauge invariance)

$$\epsilon^{(\lambda)} = \exp(-i\varphi) \left(E^{(\lambda)} - \frac{\lambda}{\sqrt{2}} \exp(i\lambda\varphi) \ \tan(\theta/2) \ (T - Z) \right) \tag{24}$$

Then, for example,

$$\tan(\theta/2) \ \exp(i\varphi) = \exp(i\varphi) \left(\theta/2 + \theta^3/24 + ... \right) = z/2 + z^2 z^\star/24 + ... \tag{25}$$

which is also analytic in z and $z^\star$.

Scalar products between 4-momenta and polarization 4-vectors can also be considered as analytic functions of z and $z^\star$. As for the scalar product of two 4-momenta p and p' it involves the scalar product of unitary 3-vectors

$$\vec{n} = (\sin(\theta) \ \cos(\varphi), \ \sin(\theta) \ \sin(\varphi), \ \cos(\theta))$$

and

$$\vec{n'} = (\sin(\theta') \ \cos(\varphi'), \ \sin(\theta') \ \sin(\varphi'), \ \cos(\theta'))$$

i.e :

$$\vec{n}.\vec{n'} = \cos(\theta) \ \cos(\theta') + \sin(\theta) \ \sin(\theta') \cos(\varphi - \varphi') \tag{26}$$

Transforming $\cos(\varphi - \varphi')$ into the sum of exponentials $\exp(i(\varphi - \varphi'))$ and $\exp(-i(\varphi - \varphi'))$ we see that such scalar products involve combinations of

$\cos\theta$, $\cos\theta'$, $\sin\theta \ \exp(\pm i\varphi)$ and $\sin\theta' \ \exp(\pm i\varphi')$

We can write

$$\begin{aligned}
\cos\theta &= 1 - zz^\star/2 + ... \\
\cos\theta' &= 1 - z'z'^\star/2 + ... \\
\sin\theta \ \exp(i\varphi) &= z - z^2 z^\star/6 + ... \\
\sin\theta \ \exp(-i\varphi) &= z^\star - z z^{\star 2}/6 + ... \\
\sin\theta' \ \exp(i\varphi') &= z' - z'^2 z'^\star/6 + ... \\
\sin\theta' \ \exp(-i\varphi') &= z'^\star - z' z'^{\star 2}/6 + ...
\end{aligned} \tag{27}$$

which demonstrates their analyticity. A scalar product of a 4-momentum p_n and a polarization 4-vector of a final photon is written as

$$\epsilon^{(\lambda)\star}.p_n = \frac{\lambda}{\sqrt{2}} \left(p_{nT} \exp\left(i\lambda[\phi - \phi_n] \right) - p_{n+} \tan(\theta/2) \right) \tag{28}$$

$$= \frac{\lambda}{\sqrt{2}} \exp(i\lambda\phi) \left(|\vec{p_n}| \sin\theta_n \exp(-i\lambda\phi_n) - p_{n+} \tan(\theta/2) \exp(-i\lambda\phi) \right)$$

As seen before, $\sin\theta_n\ \exp(-i\lambda\phi_n)$ and $\tan(\theta/2)\exp(-i\lambda\phi)$ are analytic functions respectively, of z_n and $z_n^\star$, and of z_γ and $z_\gamma^\star$. So, such scalar products are analytic functions of the z and z' involved and of their conjugates. This, in fact was to be expected since scalar products of ordinary 3-vectors are constructed from tensorial products of components of spinors pertaining to the fundamental representation of spin 1/2 of the rotational group. But we have seen that apart from global phase factors $\exp(i\lambda\varphi)$ that we factorize, they are analytic in the z and $z^\star$.

Finally, when multiplying all these functions we get surely an analytic expression in z and $z^\star$, and thus (the numerator of) the impact factor should be analytic in z and $z^\star$.

To illustrate this, let us consider some examples.

- Some helicity amplitudes $A(\lambda_{\gamma_1},\ \lambda_{\gamma_2},\ \lambda_{\ell_1},\ \lambda_{\ell_2})$ for the Compton effect $\ell_1+\gamma_1 \to \ell_2+\gamma_2$. Apart from coupling constant, we have

$$A(+,+,\downarrow,\uparrow) = -\exp(-i\varphi_{\gamma_1}+i\varphi_{\gamma_2}+i\varphi_{\ell_1}/2-i\varphi_{\ell_2}/2)\left(\frac{2\cos^3(\theta/2)}{1-\beta'\sin^2(\theta/2)}\right)$$

$$A(+,+,\uparrow,\downarrow,) = \exp(-i\varphi_{\gamma_1}+i\varphi_{\gamma_2}-i\varphi_{\ell_1}/2-i\varphi_{\ell_2}/2). \left(\frac{2m\exp(i\varphi_{\ell_2})\sin(\theta)\cos(\theta/2)}{\sqrt{s}(1-\beta'\sin^2(\theta/2))}\right)$$

$$A(+,-,\downarrow,\uparrow,) = -\exp(-i\varphi_{\gamma_2}-i\varphi_{\gamma_1}+i\varphi_{\ell_1}/2+i\varphi_{\ell_2}/2). \left(\frac{2m\exp(i\varphi_{\ell_2})\sin^3(\theta/2)}{\sqrt{s}(1-\beta'\sin^2(\theta/2))}\right)$$

  where θ is the emission angle of ℓ_2 and $\beta' = 1-m^2/s$. Of course, we usually take all φ's $=0$.

- Vertex pion-nucleon with a simple Yukawa coupling. Amplitudes $A(\lambda_3,\lambda_1)$ are $\propto \bar{U}_3\gamma_5\bar{U}_1$. Apart from factors depending on mass and energies and omitting denominators, we find

$$A(\uparrow,\uparrow) \sim \exp(i\varphi_3/2-i\varphi_1/2)\,(\cos(\theta_1/2)\cos(\theta_3/2)+ \sin(\theta_1/2)\exp(-i\varphi_1)\sin(\theta_1/2)\exp(-i\varphi_3))$$

$$A(\uparrow,\downarrow) \sim \exp(i\varphi_3/2+i\varphi_1/2)\,(\cos(\theta_1/2)\sin(\theta_3/2)\exp(-i\varphi_3)- \cos(\theta_3/2)\sin(\theta_1/2)\exp(-i\varphi_1))$$

- Pion-Nucleon scattering with Yukawa coupling $\Pi(q)+N(p_1)\to\Pi(k)+N(p_3)$. A very simple calculation shows that the numerator of the amplitude is proportional to $\bar{U}_3\hat{k}\bar{U}_1$.

$$A(\uparrow,\uparrow) \sim \exp(i\varphi_3/2-i\varphi_1/2)\,(\exp(\alpha_1/2+\alpha_3/2)\,(\cos(\theta_3/2)k_- - \sin(\theta_3/2)\exp(-i\varphi_3)(k_x+ik_y) + \exp(-\alpha_1/2-\alpha_3/2)\,(\cos(\theta_3/2)k_+ + \sin(\theta_3/2)\exp(-i\varphi_3)(k_x+ik_y)))$$

with $k_+ = k_0 + k_z$, $k_- = (m_\pi^2 + k_T^2)/k_+$, $\theta_1 = 0$ (with $\theta_1 \neq 0$, we get a longer result but the general structure remains the same).

From these examples, we see that once the phase factor $\exp\left(i \sum (\lambda_{out}\varphi_{out} - \lambda_{in}\varphi_{in})\right)$ has been extracted, the remaining factor is indeed a functional of the z's and $z^\star$'s of all the particles entering into the reaction, through simple trigonometric functions that admit a Taylor expansion relatively to z and $z^\star$'. We may consider this as a very general result, independent on the process and on energies as well, that could be ascribed to the way we calculate transition amplitudes in the framework of Quantum Field theory.

So, as regards angular variables, impact factors take on the general form

$$J \sim \exp\Big(i \sum (\lambda_{out}\varphi_{out} - \lambda_{in}\varphi_{in})\Big) \;\; F(\{z\}\,, \{z^\star\}) \tag{29}$$

As is expected from the common belief in analyticity of S-matrix elements, the function F can be expanded in powers, with positive integer exponents, of z and $z^\star$, at least in the domain $|z| \ll 1$. Indeed, this appears to be the case for all the F we are dealing with. This is a sufficient condition for proving the helicity properties of impact factors found in section 2.

4 Exact results about the strict forward direction where all $|z|$'s = 0

As a general remark, we remind that denominators of propagators are left apart from the discussion.

4.1 The subprocess $\ell_1(p_1) + \gamma^\star(q) \to \ell_3(p_3) + (N-1)$real photons$(k_j)$

4.1.1 Impact factors

Impact factors are projection onto the light-like 4-vector $T - Z$. A generic form for numerators of these impact factors is

$$\mathcal{N} = \bar{U}_3 A \gamma(T - Z) B U_1 \tag{30}$$

where the matrix B (B for before) is given by

$$B = [m + \gamma(p_1 - k_1 - k_2 - \cdots - k_{p-1})]\gamma(\epsilon^\star_{p-1})[m + \gamma(p_1 - k_1 - k_2 - \cdots \\ - k_{p-2})]\gamma(\epsilon^\star_{p-2}) \cdots [m + \gamma(p_1 - k_1)]\gamma(\epsilon^\star_1) \tag{31}$$

and the matrix A (A for after) given by

$$A = \gamma(\epsilon^\star_N)[m + \gamma(p_3 + k_N)]\gamma(\epsilon^\star_{N-1})[m + \gamma(p_3 + k_N + k_{N-1})]\gamma(\epsilon^\star_{N-2}) \cdots \\ \times \gamma(\epsilon^\star_{p+1})[m + \gamma(p_3 + k_N + k_{N-1} + \cdots + k_{p+1})] \tag{32}$$

where the notations $\gamma(v) = \gamma_\mu v^\mu$ and $k_p = -q$ (q is the 4-momentum of the virtual photon) have been used. In the above amplitude, the vertex with the virtual photon is inserted at the p-th rank, after the emission of $n_\gamma^b = p-1$ real photons (b for before) and before the emission of $n_\gamma^a = N-p$ (a for after) real photons, from the lepton line. Let us also denote by $n_\gamma = n_\gamma^b + n_\gamma^a = N-1$ the total number of real photons emitted.

For the forward direction, we have

$$\begin{gathered} p_3 \equiv \mathring{p}_3 = E_3 T + |\vec{p}_3| Z \\ p_1 \equiv \mathring{p}_1 = E_1 T + |\vec{p}_1| Z \quad \text{(this last one is exact anyway)} \\ k_j \equiv \mathring{k}_j = \omega_j (T+Z) \\ \epsilon_j^\star \equiv E_j^\star \equiv \exp(i\lambda_j\varphi_j) E^{(\lambda_j)\star} \quad \text{with} \quad E^{(\pm)} = \mp\frac{1}{\sqrt{2}}(X \pm iY) \end{gathered} \tag{33}$$

and the generic form $\mathcal{N}$ becomes

$$\mathcal{N} = \bar{\mathring{U}}_3 \; \mathring{A} \, \gamma(T-Z) \; \mathring{B} \mathring{U}_1 \tag{34}$$

where the symbol "$\circ$" above letters means that we are taking all quantities for all θ's $=0$. Using Dirac equation and the above forms (33) of 4-vectors we get

$$\begin{aligned} &\bar{\mathring{U}}_3 \gamma(E_N^\star)[m + \gamma(\mathring{p}_3 + \mathring{k}_N)]\gamma(E_{N-1}^\star)[m + \gamma(\mathring{p}_3 + \mathring{k}_N + \mathring{k}_{N-1})] \\ &= \bar{\mathring{U}}_3 \gamma(E_N^\star)\gamma(\mathring{k}_N)\gamma(E_{N-1}^\star)[m + \gamma(\mathring{p}_3 + \mathring{k}_N + \mathring{k}_{N-1})] \\ &= \bar{\mathring{U}}_3 \gamma(E_N^\star)\gamma(\mathring{k}_N)\gamma(E_{N-1}^\star)[m + \gamma(\mathring{p}_3)] = -2\, \mathring{p}_3 \cdot \mathring{k}_N \; \bar{\mathring{U}}_3 \gamma(E_N^\star)\gamma(E_{N-1}^\star) \end{aligned} \tag{35}$$

Next,

$$\begin{aligned} &\bar{\mathring{U}}_3 \gamma(E_N^\star)\gamma(E_{N-1}^\star)\gamma(E_{N-2}^\star)[m + \gamma(\mathring{p}_3 + \mathring{k}_N + \mathring{k}_{N-1} + \mathring{k}_{N-2})]\gamma(E_{N-3}^\star) \\ &\times[m + \gamma(\mathring{p}_3 + \mathring{k}_N + \mathring{k}_{N-1} + \mathring{k}_{N-2} + \mathring{k}_{N-3})] \equiv \bar{\mathring{U}}_3 \gamma(E_N^\star)\gamma(E_{N-1}^\star)\gamma(E_{N-2}^\star) \\ &\times\gamma(\mathring{k}_N + \mathring{k}_{N-1} + \mathring{k}_{N-2})\gamma(E_{N-3}^\star)[m + \gamma(\mathring{p}_3)] \\ &= -2\, \mathring{p}_3 \cdot (\mathring{k}_N + \mathring{k}_{N-1} + \mathring{k}_{N-2}) \, \bar{\mathring{U}}_3 \gamma(E_N^\star)\gamma(E_{N-1}^\star)\gamma(E_{N-2}^\star)\gamma(E_{N-3}^\star) \end{aligned} \tag{36}$$

and so on. Then, either $n_\gamma^a = N-p$ is even and we can go on the reduction until ($N-p-1$ is odd)

$$\begin{aligned} \bar{\mathring{U}}_3 \; \mathring{A} \equiv\, & (-2\, \mathring{p}_3 \cdot \mathring{k}_N)(-2\, \mathring{p}_3 \cdot (\mathring{k}_N + \mathring{k}_{N-1} + \mathring{k}_{N-2})) \cdots (-2\, \mathring{p}_3 \cdot (\mathring{k}_N + \mathring{k}_{N-1} + \\ & \mathring{k}_{N-2} + \cdots + \mathring{k}_{p+2})) \; \bar{\mathring{U}}_3 \gamma(E_N^\star)\gamma(E_{N-1}^\star)\gamma(E_{N-2}^\star) \cdots \gamma(E_{p+1}^\star) \end{aligned} \tag{37}$$

or n_γ^a is odd ($N-p-1$ is even) and we must stop the reduction at

$$\bar{\overset{\circ}{U}}_3 \overset{\circ}{A} \equiv (-2 \overset{\circ}{p}_3 \cdot \overset{\circ}{k}_N)(-2 \overset{\circ}{p}_3 \cdot (\overset{\circ}{k}_N + \overset{\circ}{k}_{N-1} + \overset{\circ}{k}_{N-2})) \cdots (-2 \overset{\circ}{p}_3 \cdot (\overset{\circ}{k}_N + \overset{\circ}{k}_{N-1} + \\ + \overset{\circ}{k}_{N-2} + \cdots + \overset{\circ}{k}_{p+3})) \bar{\overset{\circ}{U}}_3 \gamma(E^\star_N)\gamma(E^\star_{N-1})\gamma(E^\star_{N-2}) \cdots \gamma(E^\star_{p+1}) \\ \times \gamma(\overset{\circ}{k}_N + \overset{\circ}{k}_{N-1} + \cdots + \overset{\circ}{k}_{p+1}) \tag{38}$$

On the side of the incoming lepton, we can perform the same kind of reduction. We thus get

$$\overset{\circ}{B}\overset{\circ}{U}_1 \equiv (2 \overset{\circ}{p}_1 \cdot \overset{\circ}{k}_1)(2 \overset{\circ}{p}_1 \cdot (\overset{\circ}{k}_1 + \overset{\circ}{k}_2 + \overset{\circ}{k}_3)) \cdots (2 \overset{\circ}{p}_1 \cdot (\overset{\circ}{k}_1 + \overset{\circ}{k}_2 + \cdots + \overset{\circ}{k}_{p-2})) \\ \times \gamma(E^\star_{p-1})\gamma(E^\star_{p-2}) \cdots \gamma(E^\star_1) \overset{\circ}{U}_1 \tag{39}$$

if $n^b_\gamma = p - 1$ is even and

$$\overset{\circ}{B}\overset{\circ}{U}_1 \equiv -(2 \overset{\circ}{p}_1 \cdot \overset{\circ}{k}_1)(2 \overset{\circ}{p}_1 \cdot (\overset{\circ}{k}_1 + \overset{\circ}{k}_2 + \overset{\circ}{k}_3)) \cdots (2 \overset{\circ}{p}_1 \cdot (\overset{\circ}{k}_1 + \overset{\circ}{k}_2 + \cdots + \overset{\circ}{k}_{p-3})) \\ \times \gamma(\overset{\circ}{k}_1 + \overset{\circ}{k}_2 + \cdots + \overset{\circ}{k}_{p-1})\gamma(E^\star_{p-1})\gamma(E^\star_{p-2}) \cdots \gamma(E^\star_1) \overset{\circ}{U}_1 \tag{40}$$

if $n^b_\gamma = p - 1$ is odd.

Let us now consider separately the cases where n_γ is even or odd.

1) $\underline{n_\gamma \text{ is odd.}}$

We have then the only two possibilities : either n^a_γ is even and n^b_γ is odd, or n^a_γ is odd and n^b_γ is even. Accordingly, we are led to expressions like

$$(-2 \overset{\circ}{p}_3 \cdot \overset{\circ}{k}_N)(-2 \overset{\circ}{p}_3 \cdot (\overset{\circ}{k}_N + \overset{\circ}{k}_{N-1} + \overset{\circ}{k}_{N-2})) \cdots (-2 \overset{\circ}{p}_3 \cdot (\overset{\circ}{k}_N + \overset{\circ}{k}_{N-1} + \\ + \overset{\circ}{k}_{N-2} + \cdots + \overset{\circ}{k}_{p+2})) \\ \times \bar{\overset{\circ}{U}}_3 \gamma(E^\star_N)\gamma(E^\star_{N-1})\gamma(E^\star_{N-2}) \cdots \gamma(E^\star_{p+1})\gamma(T - Z) \\ \gamma(\overset{\circ}{k}_1 + \overset{\circ}{k}_2 + \cdots + \overset{\circ}{k}_{p-1})\gamma(E^\star_{p-1})\gamma(E^\star_{p-2}) \cdots \gamma(E^\star_1) \overset{\circ}{U}_1 \\ \times (-)(2 \overset{\circ}{p}_1 \cdot \overset{\circ}{k}_1)(2 \overset{\circ}{p}_1 \cdot (\overset{\circ}{k}_1 + \overset{\circ}{k}_2 + \overset{\circ}{k}_3)) \cdots (2 \overset{\circ}{p}_1 \cdot (\overset{\circ}{k}_1 + \overset{\circ}{k}_2 + \cdots + \overset{\circ}{k}_{p-3})) \tag{41}$$

for n^a_γ even and n^b_γ odd, and

$$(-2 \overset{\circ}{p}_3 \cdot \overset{\circ}{k}_N)(-2 \overset{\circ}{p}_3 \cdot (\overset{\circ}{k}_N + \overset{\circ}{k}_{N-1} + \overset{\circ}{k}_{N-2})) \cdots (-2 \overset{\circ}{p}_3 \cdot (\overset{\circ}{k}_N + \overset{\circ}{k}_{N-1} + \\ + \overset{\circ}{k}_{N-2} + \cdots + \overset{\circ}{k}_{p+3})) \\ \bar{\overset{\circ}{U}}_3 \gamma(E^\star_N)\gamma(E^\star_{N-1})\gamma(E^\star_{N-2}) \cdots \gamma(E^\star_{p+1})\gamma(\overset{\circ}{k}_N + \overset{\circ}{k}_{N-1} + \cdots + \overset{\circ}{k}_{p+1}) \\ \times \gamma(T - Z)\gamma(E^\star_{p-1})\gamma(E^\star_{p-2}) \cdots \gamma(E^\star_1) \overset{\circ}{U}_1 \\ \times (2 \overset{\circ}{p}_1 \cdot \overset{\circ}{k}_1)(2 \overset{\circ}{p}_1 \cdot (\overset{\circ}{k}_1 + \overset{\circ}{k}_2 + \overset{\circ}{k}_3)) \cdots (2 \overset{\circ}{p}_1 \cdot (\overset{\circ}{k}_1 + \overset{\circ}{k}_2 + \cdots + \overset{\circ}{k}_{p-2})) \tag{42}$$

for n_γ^a odd and n_γ^b even. But

$$\gamma(\overset{\circ}{k}_1 + \overset{\circ}{k}_2 + \cdots + \overset{\circ}{k}_{p-1}) = (\omega_1 + \cdots + \omega_{p-1})\gamma(T+Z) \tag{43}$$

and

$$\gamma(\overset{\circ}{k}_N + \overset{\circ}{k}_{N-1} + \cdots + \overset{\circ}{k}_{p+1}) = (\omega_N + \cdots + \omega_{p+1})\gamma(T+Z) \tag{44}$$

Thus the generic form should be proportional to

$$\bar{\overset{\circ}{U}}_3\gamma(E_N^\star)\cdots\gamma(E_{p+1}^\star)\gamma(E_{p-1}^\star)\cdots\gamma(E_1^\star)\gamma(T-Z)\gamma(T+Z)\,\overset{\circ}{U}_1 \tag{45}$$

in the first case or

$$\bar{\overset{\circ}{U}}_3\gamma(E_N^\star)\cdots\gamma(E_{p+1}^\star)\gamma(E_{p-1}^\star)\cdots\gamma(E_1^\star)\gamma(T+Z)\gamma(T-Z)\,\overset{\circ}{U}_1 \tag{46}$$

in the second case, or, equivalently, proportional to

$$\bar{\overset{\circ}{U}}_3\gamma(E_N^\star)\cdots\gamma(E_{p+1}^\star)\gamma(E_{p-1}^\star)\cdots\gamma(E_1^\star)\,(1 \mp 2\lambda_1\gamma_5)\,\overset{\circ}{U}_1 \tag{47}$$

Such an expression is zero if

- two successive $E_j^\star$ are the same
- $\lambda_{\gamma_1} = -2\lambda_1$

Thus, in order to obtain a non-zero result, we must have

$$\lambda_{\gamma_1} = -\lambda_{\gamma_2} = \lambda_{\gamma_3} = \cdots = \lambda_{\gamma_{N-2}} = -\lambda_{\gamma_{N-1}} = 2\lambda_1 \tag{48}$$

and the studied term is then proportional to

$$\bar{\overset{\circ}{U}}_3\gamma(E_N^\star)\,(1 \mp 2\lambda_1\gamma_5)\,\overset{\circ}{U}_1 \tag{49}$$

which is non-zero only for $\lambda_{\gamma_N} = -2\lambda_3 = 2\lambda_1$. As a conclusion, when n_γ is odd, only HNC amplitudes have non-zero values[3]. This was to be expected from the general rule (18) : $\Lambda + \lambda_3 - \lambda_1 = n - p$ that here gives $(n = p = 0)$ $\Lambda = \lambda_1 - \lambda_3$. But when n_γ is odd, Λ is necessarily odd and so should also be the difference $\lambda_1 - \lambda_3$. Obviously, this can be achieved only for HNC amplitudes.

Moreover, it can be easily shown that (49) is proportional to the lepton mass m[4]. As for scalar products of 4-vectors appearing in (41) or (42) we have, for example,

$$\overset{\circ}{p}_3 \cdot \overset{\circ}{k}_j = \omega_j\,\overset{\circ}{p}_{3-} = m^2\omega_j/\,\overset{\circ}{p}_{3+} \quad , \quad \overset{\circ}{p}_1 \cdot \overset{\circ}{k}_r = m^2\omega_r/\,\overset{\circ}{p}_{1+} \tag{50}$$

[3]HNC means : helicity non-conserving, and HC : helicity conserving
[4]Spinors are normalized according to $\bar{U}U = 2m$

so that the resulting amplitude (41) is proportional to

$$(m^2)^{(N-p)/2}(m^2)^{(p-2)/2}m = m^{n_\gamma} \tag{51}$$

It is easy to show that the same result applies for the amplitude (42). For instance, for $n_\gamma = 1$, only the spin-flip amplitude is a priori non-zero for the forward direction and its numerator should be $\propto m$, which is indeed the case. We also notice that we here find again that HNC amplitudes should be odd functions of m.

2) $\underline{n_\gamma \text{ is even.}}$

In this case, we have the two possibilities : n_γ^a and n_γ^b both even or n_γ^a and n_γ^b both odd.

a) n_γ^a and n_γ^b both even.

The generic form is then

$$\begin{gathered}(-2\,\mathring{p}_3\cdot\mathring{k}_N)\cdots(-2\,\mathring{p}_3\cdot(\mathring{k}_N+\mathring{k}_{N-1}++\cdots+\mathring{k}_{p+2}))\\ \times\bar{\mathring{U}}_3\gamma(E_N^\star)\cdots\gamma(E_{p+1}^\star)\gamma(T-Z)\gamma(E_{p-1}^\star)\cdots\gamma(E_1^\star)\,\mathring{U}_1\\ \times(2\,\mathring{p}_1\cdot\mathring{k}_1)\cdots(2\,\mathring{p}_1\cdot(\mathring{k}_1+\mathring{k}_2+\cdots+\mathring{k}_{p-2}))\end{gathered} \tag{52}$$

The above matrix element is non-zero only if

$$\lambda_{\gamma_1} = 2\lambda_1 = -\lambda_{\gamma_2} = \lambda_{\gamma_3} = \cdots = \lambda_{\gamma_{N-1}} = -\lambda_{\gamma_N} = 2\lambda_3 \tag{53}$$

and thus, all analogous amplitudes are non-zero if

$$\sum_{j\neq p}\lambda_{\gamma_j} = 0 \qquad \text{and} \qquad \lambda_1 = \lambda_3 \tag{54}$$

Under these conditions, the matrix element is proportional to the following one

$$\bar{\mathring{U}}_3(\lambda_1)\gamma(T-Z)\,\mathring{U}_1\,(\lambda_1) \tag{55}$$

which can be shown to be of order zero in m. Consequently, this case a) corresponds to terms in <u>HC amplitudes</u> that are proportional to

$$(m^2)^{(N-p)/2}(m^2)^{(p-1)/2}m = m^{n_\gamma} \tag{56}$$

b) n_γ^a and n_γ^b both odd.

Now, the generic form is

$$\begin{gathered}(-2\,\mathring{p}_3\cdot\mathring{k}_N)\cdots(-2\,\mathring{p}_3\cdot(\mathring{k}_N+\mathring{k}_{N-1}++\cdots+\mathring{k}_{p+3}))\\ \times\bar{\mathring{U}}_3\gamma(E_N^\star)\cdots\gamma(E_{p+1}^\star)\gamma(\mathring{k}_N+\mathring{k}_{N-1}+\cdots+\mathring{k}_{p+1})\\ \gamma(T-Z)\gamma(\mathring{k}_1+\mathring{k}_2+\cdots+\mathring{k}_{p-1})\gamma(E_{p-1}^\star)\cdots\gamma(E_1^\star)\,\mathring{U}_1\\ \times(-)(2\,\mathring{p}_1\cdot\mathring{k}_1)\cdots(2\,\mathring{p}_1\cdot(\mathring{k}_1+\mathring{k}_2+\cdots+\mathring{k}_{p-3}))\end{gathered} \tag{57}$$

But because of (42), (43), (44) and since $\gamma(T+Z)\gamma(T-Z)\gamma(T+Z) = 2\gamma(T+Z)$, this amplitude is proportional to

$$\bar{\mathring{U}}_3\gamma(E^\star_N)\cdots\gamma(E^\star_{p+1})\gamma(E^\star_{p-1})\cdots\gamma(E^\star_1)\gamma(T+Z)\,\mathring{U}_1 \tag{58}$$

Here again, we find that this term is non-zero only if conditions (54) are satisfied, and we are led to the simple matrix element

$$\bar{\mathring{U}}_3(\lambda_1)\gamma(T+Z)\,\mathring{U}_1\,(\lambda_1) \tag{59}$$

which is proportional to m^2. Therefore, the terms corresponding to that case b) are associated with HC amplitudes and are proportional to

$$(m^2)^{(N-p-1)/2}(m^2)^{(p-2)/2}m^2 = m^{n_\gamma} \tag{60}$$

So, as a general conclusion for the subprocess under study, we may state that impact factors in the forward direction are all proportional to m^{n_γ}. Moreover, in this limit, only HC amplitudes survive if n_γ is even while only HNC amplitudes survive if n_γ is odd. This is in complete agreement with the above-mentioned general rule $\Lambda + \lambda_3 - \lambda_1 = n - p$ for $n = p = 0$:

- if n_γ is even, Λ is even and we must have $\lambda_3 = \lambda_1$, and then $\Lambda = 0$;
- if n_γ is odd, Λ is odd and we must have $\lambda_3 = -\lambda_1$, $\Lambda = 2\lambda_1$.

4.1.2 Other projections

1) J_- components

This component is the projection of the current onto $T+Z$. It is clear from (43), (44) and (57) that replacing $T-Z$ by $T+Z$ yields a zero result. Thus, as far as numerators are concerned, the component J_- is strictly zero in the forward direction configuration if n_γ is odd. If n_γ is even, we a priori get a non-zero result if n^a_γ and n^b_γ are both even. This concerns only HC amplitudes and the corresponding amplitude is proportional to (59) which itself is $\propto m^2$. Finally, J_- is proportional to $m^{n_\gamma+2}$. A similar result may be found in another context.

The relation that expresses current conservation

$$q^\mu J_\mu = 0 = J_- q_+/2 + J_+ q_-/2 - \vec{q}_T \cdot \vec{J}_T \tag{61}$$

for the current J describing the subprocess, would give, for the forward direction (keeping a priori denominators unchanged...),

$$J_- = J\cdot(T+Z) = -q_- J_+/q_+ \tag{62}$$

But in this limit $q_- = \mathring{p}_{3-} - \mathring{p}_{1-} = m^2(1/\mathring{p}_{3+} - 1/\mathring{p}_{1+})$. Thus, we find that in this configuration

$$J_-/J_+ \propto m^2/s \tag{63}$$

for any set of helicities.

However, one should be aware of the fact that current conservation involves the whole set of factors in the current, not only numerators but denominators as well. Thus, taking $|\vec{q}_T| = 0$ in (61) certainly involves approximate forms of denominators. This explains why, for instance, in the former derivation (where denominators are left aside) we got a zero value for J_- when n_γ is odd while J_+ is not zero in that case : this means that in the kind of approximation we have in mind where denominators are kept more or less "exact", formula (61) should be considered with great care.

2) J_T components

Transverse components are projections onto $E^{(\pm)}$. Here too, we will discuss separately the cases n_γ even and n_γ odd.

a) n_γ even

Since $\gamma(T+Z)\gamma(E^{(\pm)})\gamma(T+Z) = 0$, the case "$n_\gamma^a$ odd n_γ^b odd" is excluded. For the case "n_γ^a even n_γ^b even", the generic matrix element is analogous to (52) with $E^{(\pm)\star}$ replacing $T - Z$. Then, we should have

$$\lambda_{\gamma_1} = 2\lambda_1 = -\lambda_{\gamma_2} = \lambda_{\gamma_3} = \cdots = \lambda_p \text{ (virtual photon)} = \\ \cdots = -\lambda_{\gamma_{N-1}} = \lambda_{\gamma_N} = -2\lambda_3 \tag{64}$$

This case is thus associated with HNC amplitudes and the remaining matrix element is

$$\bar{\mathring{U}}_3(-\lambda_1)\gamma(E_N^\star)\,\mathring{U}_1\,(\lambda_1) \tag{65}$$

which can be easily found to be $\propto m$. In conclusion, if n_γ is even, only HNC amplitudes survive and are $\propto m^{n_\gamma+1}$. Moreover, we have the correlation $\lambda_p = 2\lambda_1 = -2\lambda_3$.

b) n_γ odd

Let us consider the case where n_γ^a is even and n_γ^b odd. Then, if we adapt (41) to this case, the generic matrix element is

$$\begin{gathered}(-2\,\mathring{p}_3\cdot\mathring{k}_N)\cdots(-2\,\mathring{p}_3\cdot(\mathring{k}_N+\mathring{k}_{N-1}+\cdots+\mathring{k}_{p+2}))\\ \times\bar{\mathring{U}}_3\gamma(E_N^\star)\cdots\gamma(E_{p+1}^\star)\gamma(E_p^\star)\gamma(E_{p-1}^\star)\cdots\gamma(E_1^\star)\gamma(T+Z)\,\mathring{U}_1\\ \times(-)(2\,\mathring{p}_1\cdot\mathring{k}_1)\cdots(2\,\mathring{p}_1\cdot(\mathring{k}_1+\mathring{k}_2+\cdots+\mathring{k}_{p-3}))\end{gathered} \tag{66}$$

This matrix element is non-zero only if

$$\lambda_{\gamma_1} = 2\lambda_1 = -\lambda_{\gamma_2} = \cdots = -\lambda_p = \cdots = \lambda_{\gamma_{N-1}} = -\lambda_{\gamma_N} = 2\lambda_3 \tag{67}$$

It thus concerns only HC amplitudes and after reduction, the remaining matrix element is proportional to (59). We can conclude that when n_γ is odd, only HC amplitudes are non-zero for the forward direction and that they are proportional to

$$(m^2)^{(N-p)/2}(m^2)^{(p-2)/2}m^2 = m^{n_\gamma+1} \tag{68}$$

So, as a general conclusion for this subprocess, we may state that the transverse components of its current for the forward direction are $\propto m^{n_\gamma+1}$. Moreover, if n_γ is even only HNC amplitudes survive (and $\lambda_p = 2\lambda_1$) while if n_γ is odd only HC amplitudes survive (and $\lambda_p = -2\lambda_1$).

Notice that some of these results could be derived from the general constraint $\Lambda+\lambda_p = \lambda_1 - \lambda_3$ where the helicity $\pm$ of the virtual photon should be now included.

Indeed, if n_γ is even, $\Lambda + \lambda_p$ is odd, and we must have $\lambda_1 = -\lambda_3$ (HNC amplitudes). But HNC amplitudes are odd functions of m. So, we may expect that they should involve an additional power of m as compared to the dominant amplitudes which are then HC amplitudes. On the other hand, if n_γ is odd, $\Lambda+\lambda_p$ is even and then $\lambda_1 = \lambda_3$ (HC amplitudes). But HC amplitudes are even functions of m. So, we may expect that they should involve an additional power of m as compared to the dominant amplitudes which are then HNC amplitudes.

4.2 The subprocess $\gamma^\star(q) \to \bar{\ell}(p_1)+\ell(p_3)+(N-1)$real photons$(k_j)$

Let us now consider the process $\gamma^\star \to \bar{\ell} + \ell + (N-1)\ \gamma's$ and let us try to derive, as before, some properties of its current when all final particles $\ell,\ \bar{\ell},\ (N-1)\ \gamma's$ are emitted in the strict forward direction.

Here, the generic form for numerators of helicity amplitudes is

$$\begin{aligned}\mathcal{N} = \bar{U}_3\gamma(\epsilon_N^\star)[m+\gamma(p_3+k_N)]\gamma(\epsilon_{N-1}^\star)[m+\gamma(p_3+k_N+k_{N-1})]\gamma(\epsilon_{N-2}^\star)\cdots \\ \times\gamma(\epsilon_{p+1}^\star)[m+\gamma(p_3+k_N+k_{N-1}+\cdots+k_{p+1})]\ \gamma_\mu \\ \times[m-\gamma(p_1+k_1+k_2+\cdots+k_{p-1})]\gamma(\epsilon_{p-1}^\star) \\ \times[m-\gamma(p_1+k_1+k_2-\cdots+k_{p-2})]\gamma(\epsilon_{p-2}^\star)\cdots[m-\gamma(p_1+k_1)]\gamma(\epsilon_1^\star)V_{1c}\end{aligned} \tag{69}$$

where the vertex of the virtual photon stands at the p-th place (here too, we may set $q=-k_p$). The conjugate spinor V_{1c} is defined as

$$V_{1c}(\lambda_1) = -2\lambda_1\gamma_5 U_1(-\lambda_1) \tag{70}$$

Of course, the above structure is quite analogous to that studied in the preceding subsection, simply because it can be derived from the latter by crossing. So, we will apply to (1) the same reductions as those used in subsection (4.1), and we will also use similar notations.

1) $\underline{n_\gamma \text{ is odd.}}$

a) n_γ^a even and n_γ^b odd

$$(-2\overset{\circ}{p}_3\cdot\overset{\circ}{k}_N)(-2\overset{\circ}{p}_3\cdot(\overset{\circ}{k}_N+\overset{\circ}{k}_{N-1}+\overset{\circ}{k}_{N-2}))\cdots(-2\overset{\circ}{p}_3\cdot(\overset{\circ}{k}_N+\overset{\circ}{k}_{N-1}+$$

$$+ \overset{\circ}{k}_{N-2} + \cdots + \overset{\circ}{k}_{p+2})) \ \bar{\overset{\circ}{U}}_3 \ \gamma(E_N^\star)\gamma(E_{N-1}^\star)\gamma(E_{N-2}^\star)\cdots\gamma(E_{p+1}^\star) \ \gamma_\mu \qquad (71)$$
$$\gamma(\overset{\circ}{k}_1 + \overset{\circ}{k}_2 + \cdots + \overset{\circ}{k}_{p-1})\gamma(E_{p-1}^\star)\gamma(E_{p-2}^\star)\cdots\gamma(E_1^\star) \ \overset{\circ}{V}_{1c}$$
$$\times(-)(-2 \ \overset{\circ}{p}_1 \cdot \overset{\circ}{k}_1)(-2 \ \overset{\circ}{p}_1 \cdot(\overset{\circ}{k}_1 + \overset{\circ}{k}_2 + \overset{\circ}{k}_3))\cdots(-2 \ \overset{\circ}{p}_1 \cdot(\overset{\circ}{k}_1 + \overset{\circ}{k}_2 + \cdots + \overset{\circ}{k}_{p-3}))$$

b) n_γ^a odd and n_γ^b even

$$(-2 \ \overset{\circ}{p}_3 \cdot \overset{\circ}{k}_N)(-2 \ \overset{\circ}{p}_3 \cdot(\overset{\circ}{k}_N + \overset{\circ}{k}_{N-1} + \overset{\circ}{k}_{N-2}))\cdots(-2 \ \overset{\circ}{p}_3 \cdot(\overset{\circ}{k}_N + \overset{\circ}{k}_{N-1} +$$
$$+ \overset{\circ}{k}_{N-2} + \cdots + \overset{\circ}{k}_{p+3})) \ \bar{\overset{\circ}{U}}_3 \ \gamma(E_N^\star)\gamma(E_{N-1}^\star)\gamma(E_{N-2}^\star)\cdots\gamma(E_{p+1}^\star)$$
$$\times\gamma(\overset{\circ}{k}_N + \overset{\circ}{k}_{N-1} + \cdots + \overset{\circ}{k}_{p+1}) \ \gamma_\mu \ \gamma(E_{p-1}^\star)\gamma(E_{p-2}^\star)\cdots\gamma(E_1^\star) \ \overset{\circ}{V}_{1c} \qquad (72)$$
$$\times(-2 \ \overset{\circ}{p}_1 \cdot \overset{\circ}{k}_1)(-2 \ \overset{\circ}{p}_1 \cdot(\overset{\circ}{k}_1 + \overset{\circ}{k}_2 + \overset{\circ}{k}_3))\cdots(-2 \ \overset{\circ}{p}_1 \cdot(\overset{\circ}{k}_1 + \overset{\circ}{k}_2 + \cdots + \overset{\circ}{k}_{p-2}))$$

- From these two expression it is clear that when n_γ is odd, all corresponding components $J_- = J \cdot (T + Z)$ of the current are strictly zero in the forward configuration.
- Regarding impact factors $J_+ = J \cdot (T - Z)$, they are non-zero only if

$$\begin{array}{c} \lambda_{\gamma_1} = -2\lambda_1 = -\lambda_{\gamma_2} = \cdots = \lambda_{\gamma_{p-1}} = -\lambda_{\gamma_{p+1}} = \\ \cdots = -\lambda_{\gamma_{N-1}} = \lambda_{\gamma_N} = -2\lambda_3 \quad \text{case (a), or} \\ \lambda_{\gamma_1} = -2\lambda_1 = -\lambda_{\gamma_2} = \cdots = -\lambda_{\gamma_{p-1}} = \lambda_{\gamma_{p+1}} = \\ \cdots = -\lambda_{\gamma_{N-1}} = \lambda_{\gamma_N} = -2\lambda_3 \quad \text{case (b)} \end{array} \qquad (73)$$

i.e. only for HC amplitudes. In this case, we find an amplitude proportional to the matrix element

$$\bar{\overset{\circ}{U}}_3(\lambda_1)\gamma(T \mp Z)\gamma(T \pm Z)\gamma(E^{(-2\lambda_1)\star}) \ \overset{\circ}{V}_{1c} \ (\lambda_1) \qquad (74)$$

which is itself proportional to the following one

$$\bar{\overset{\circ}{U}}_3(\lambda_1)(1 \mp 2\lambda_1\gamma_5) \ \overset{\circ}{U}_1 \ (\lambda_1) \qquad (75)$$

and the latter is proportional to the lepton mass m.

Thus, we here find that only HC impact factors survive and should be proportional to

$$(m^2)^{(N-p)/2}(m^2)^{(p-2)/2}m = m^{n_\gamma} \qquad (76)$$

- The transverse components which are projections onto $E^{(\pm)} \equiv E^{(\lambda_{\gamma_p})\star}$, are a priopri non-zero only if

$$\begin{array}{c} \lambda_{\gamma_1} = -2\lambda_1 = -\lambda_{\gamma_2} = \cdots = \lambda_{\gamma_{p-1}} = -\lambda_{\gamma_p} = \lambda_{\gamma_{p+1}} = \\ \cdots = \lambda_{\gamma_{N-1}} = -\lambda_{\gamma_N} = 2\lambda_3 \quad \text{case (a)}, \quad \text{or} \\ \lambda_{\gamma_1} = -2\lambda_1 = -\lambda_{\gamma_2} = \cdots = -\lambda_{\gamma_{p-1}} = \lambda_{\gamma_p} = -\lambda_{\gamma_{p+1}} = \\ \cdots = \lambda_{\gamma_{N-1}} = -\lambda_{\gamma_N} = 2\lambda_3 \quad \text{case (b)} \end{array} \tag{77}$$

  In this case, since we have now an even number of $\gamma(E^\star)$, the amplitude is finally proportional to the matrix element

$$\bar{\mathring{U}}_3(\lambda_1)\gamma(T+Z)\,\mathring{U}_1\,(\lambda_1) \tag{78}$$

  which is itself proportional to m^2. Thus, only HNC transverse components survive and they are proportional to $\underline{m^{n_\gamma+1}}$.

It is worth noticing that "mass-parity" seems here different to that in subsection (4.1). But this is simply due to the choice we made to define the helicity of the antilepton. With the definition (70), amplitudes are proportional to the phase factor $\exp(i\lambda_1\varphi_1)$ (as if we had an outgoing lepton), and the general rule $\Lambda + \lambda_3 + \lambda_1 = n - p$ should be applied. Taking $n = p = 0$, we get, when n_γ is odd :

- $\lambda_3 = \lambda_1$ for impact factor, because then Λ, which is the sum of the helicities of the outgoing photons, is odd ; hence, HC amplitudes ;
- $\lambda_3 = -\lambda_1$ for transverse components, because Λ, which should include the ± 1 helicity of the virtual photon, is now even ; hence, HNC amplitudes.

2) $\underline{n_\gamma}$ is even.

a) n_γ^a and n_γ^b are even

$$\begin{array}{c} (-2\,\mathring{p}_3\cdot\mathring{k}_N)(-2\,\mathring{p}_3\cdot(\mathring{k}_N+\mathring{k}_{N-1}+\mathring{k}_{N-2}))\cdots(-2\,\mathring{p}_3\cdot(\mathring{k}_N+\mathring{k}_{N-1}+ \\ +\mathring{k}_{N-2}+\cdots+\mathring{k}_{p+2}))\ \bar{\mathring{U}}_3\ \gamma(E_N^\star)\gamma(E_{N-1}^\star)\gamma(E_{N-2}^\star)\cdots\gamma(E_{p+1}^\star)\ \gamma_\mu \\ \gamma(E_{p-1}^\star)\gamma(E_{p-2}^\star)\cdots\gamma(E_1^\star)\ \mathring{V}_{1c} \\ \times(-2\,\mathring{p}_1\cdot\mathring{k}_1)(-2\,\mathring{p}_1\cdot(\mathring{k}_1+\mathring{k}_2+\mathring{k}_3))\cdots(-2\,\mathring{p}_1\cdot(\mathring{k}_1+\mathring{k}_2+\cdots+\mathring{k}_{p-2})) \end{array} \tag{79}$$

b) n_γ^a and n_γ^b are odd

$$\begin{array}{c} (-2\,\mathring{p}_3\cdot\mathring{k}_N)(-2\,\mathring{p}_3\cdot(\mathring{k}_N+\mathring{k}_{N-1}+\mathring{k}_{N-2}))\cdots(-2\,\mathring{p}_3\cdot(\mathring{k}_N+\mathring{k}_{N-1}+ \\ +\mathring{k}_{N-2}+\cdots+\mathring{k}_{p+3}))\ \bar{\mathring{U}}_3\ \gamma(E_N^\star)\gamma(E_{N-1}^\star)\gamma(E_{N-2}^\star)\cdots\gamma(E_{p+1}^\star) \\ \times\gamma(\mathring{k}_N+\mathring{k}_{N-1}+\cdots+\mathring{k}_{p+1})\ \gamma_\mu\ \gamma(\mathring{k}_1+\mathring{k}_2+\cdots+\mathring{k}_{p-1}) \\ \times\gamma(E_{p-1}^\star)\gamma(E_{p-2}^\star)\cdots\gamma(E_1^\star)\ \mathring{V}_{1c}\times(-)(-2\,\mathring{p}_1\cdot\mathring{k}_1) \\ \times(-2\,\mathring{p}_1\cdot(\mathring{k}_1+\mathring{k}_2+\mathring{k}_3))\cdots(-2\,\mathring{p}_1\cdot(\mathring{k}_1+\mathring{k}_2+\cdots+\mathring{k}_{p-3})) \end{array} \tag{80}$$

- J_- components exist only in case a). We must have

$$\lambda_{\gamma_1} = -2\lambda_1 = -\lambda_{\gamma_2} = \cdots = -\lambda_{\gamma_{p-1}} = \\ \lambda_{\gamma_{p+1}} = \cdots = \lambda_{\gamma_{N-1}} = -\lambda_{\gamma_N} = 2\lambda_3 \tag{81}$$

  i.e. HNC amplitudes (in agreement with the general rule) that are proportional to the matrix element

$$\bar{\overset{\circ}{U}}_3(-\lambda_1)\gamma(T+Z)\gamma_5 \overset{\circ}{U}_1 (-\lambda_1) \tag{82}$$

  which is proportional to m^2. Thus, in this case HNC J_-'s are $\underline{\propto m^{n_\gamma+2}}$.

- In both cases, impact factors are also found as HNC amplitudes (even number of $\gamma(E^\star)$'s). In case a) they are proportional to

$$\bar{\overset{\circ}{U}}_3(-\lambda_1)\gamma(T-Z)\gamma_5 \overset{\circ}{U}_1 (-\lambda_1) \tag{83}$$

  which is of order zero in m, while in case b) they are proportional to

$$\bar{\overset{\circ}{U}}_3(-\lambda_1)\gamma(T+Z)\gamma_5 \overset{\circ}{U}_1 (-\lambda_1) \tag{84}$$

  which is $\propto m^2$. Thus, in both cases, we find HNC impact factors $\underline{\propto m^{n_\gamma}}$.

- Finally, transverse components are HC amplitudes and it is not difficult to show that they are zero in case b) and $\underline{\propto m^{n_\gamma+1}}$ in case a), being $\propto$

$$\bar{\overset{\circ}{U}}_3(\lambda_1)\gamma(E^{(-2\lambda_1)\star}) \overset{\circ}{V}_{1c} (\lambda_1) \sim \bar{\overset{\circ}{U}}_3(\lambda_1) \overset{\circ}{U}_1 (\lambda_1) \propto m \tag{85}$$

5 The case of multi-bremsstrahlung processes

5.1 Introduction

Let us remind that the name "order" here refers to jet-like kinematics, where all particles are ultra-relativistic (for massive ones their energies E are $\gg m$), and all final particles of a given vertex are emitted near the direction of propagation of the parent incoming particle. Terms of order one are m/E and polar angles θ. However, in the following, by "order" of a term or of an expression we mean their "lowest order" in the jet-like kinematics approximation. Thus, $\cos(\theta)$ is said to be of order zero and $\sin(\theta)$ of order one ; $p_- = E - p_Z$ is of order two, etc.

In dealing with approximation, we have to find an efficient method providing a line of action for picking all necessary terms that yield a given order of approximation, of course without forgetting none of these terms. The problem we have in mind is the following.

Consider the generic form of an impact factor for the multi-bremsstrahlung process lepton + $\gamma^\star \to$ lepton + n_γ real photons with $n_\gamma = N-1$:

$$\begin{aligned} J = \bar{U}_3 \mathcal{M} U_1 \quad \text{with} \\ \mathcal{M} = \gamma(\epsilon^\star_N)[m+\gamma(p_3+k_N)]\gamma(\epsilon^\star_{N-1})[m+\gamma(p_3+k_N+k_{N-1})]\gamma(\epsilon^\star_{N-2})\cdots \\ \times\gamma(\epsilon^\star_{p+1})[m+\gamma(p_3+k_N+k_{N-1}+\cdots+k_{p+1})]\gamma(T-Z) \\ \times[m+\gamma(p_1-k_1-k_2-\cdots-k_{p-1})]\gamma(\epsilon^\star_{p-1})\cdots \\ \times[m+\gamma(p_1-k_1-k_2-\cdots-k_{p-2})]\gamma(\epsilon^\star_{p-2})\cdots[m+\gamma(p_1-k_1)]\gamma(\epsilon_1) \end{aligned} \tag{86}$$

where the notation $\gamma(V) = V_\mu \gamma^\mu$ is used here again and propagators have been omitted. Notice that the impact factor J in (86) has mass dimension $p-1+N-p+1 = n_\gamma+1$ (spinors being normalized according to $\bar{U}U = 2m$).

We will use the following decompositions. For any 4-momentum Q, we have

$$\begin{aligned} \gamma(Q) = \frac{Q_+}{2}\gamma(T+Z) + \frac{Q_-}{2}\gamma(T-Z) + \gamma(Q_T) \\ \text{where} \quad Q_T = Q_X X + Q_Y Y, \quad Q_\pm = Q_0 \pm Q_z \end{aligned} \tag{87}$$

and for any polarization 4-vector, we take (see (12))

$$\gamma(\epsilon^\star) = \exp(i\Lambda\varphi_\gamma)\ [\gamma(E^{(\Lambda)\star}) + \xi\ \gamma(T-Z)] \tag{88}$$

Since the leading order of $\mathcal{M}$ in (86) is $N-1 = n_\gamma$, we should find that all terms of less order in its development are strictly zero ! This can be easily checked for the term of zeroth order : let us take the leading orders in (87) and (88)

$$\gamma(Q) \approx \frac{Q_+}{2}\gamma(T+Z)\ , \quad \gamma(\epsilon^{(\Lambda)\star}) \approx \exp\,(i\Lambda\varphi_\gamma)\gamma(E^{(\Lambda)\star})$$

then

$$\begin{aligned} \mathcal{M}^{(0)} \propto \gamma(E^\star_N)\gamma(T+Z)\gamma(E^\star_{N-1})\gamma(T+Z)\cdots \\ \times\gamma(E^\star_{p+1})\gamma(T+Z)\gamma(T-Z)\gamma(T+Z)\gamma(E^\star_{p-1})\cdots\gamma(T+Z)\gamma(E^\star_1) \end{aligned} \tag{89}$$

which is identically zero since $\gamma(E^\star_k)$ and $\gamma(T+Z)$ are anticommuting and $[\gamma(T+Z)]^2 = 0$. So, to pick terms of order n_γ is not so trivial as we are forced to make the development of the product of $N-1$ factors, losing then transparency in calculation : having n_γ factors F_k that we expand according various orders in the form

$$F_k = F_k^{(0)} + F_k^{(1)} + F_k^{(2)} + \cdots$$

$\mathcal{M} = F_1 F_2 F_3 \cdots F_{n_\gamma}$ should have the leading form

$$\mathcal{M} \simeq \sum_{n_1+n_2+\cdots n_{n_\gamma}=n_\gamma} F_1^{(n_1)} F_2^{(n_2)} \cdots F_{n_\gamma}^{(n_{n_\gamma})}$$

This shows that obtaining the lowest order expression of an impact factor is generally not an easy task, and, as in any computation of approximate forms, the legitime questions arise : how do the various terms combine so as to yield the lowest-order expression ? Is it possible to discard a priori some terms ?

Our goal is then twofold. First, it is highly desirable to have a full control over the orders of the various terms entering a given transition amplitude, with the aim to make coherent the approximations one is led to perform in the framework of jet-like kinematics. Secondly, we would like to find some systematics for easy computation, if possible. In the following, we attempt to get some clues to this problem, and to develop a formalism that could be useful for computational purposes.

5.2 Tracking orders in a computer's dream

Let us consider a matrix of the form

$$\mathcal{N} = [m + \gamma(Q)] \;\; \gamma(\epsilon^\star) \tag{90}$$

Using the above decompositions (87) and (88), we get

$$\begin{gathered}
\mathcal{N} = \left[m + \frac{Q_+}{2}\gamma(T+Z) + \frac{Q_-}{2}\gamma(T-Z) + \gamma(Q_T)\right] \;\; \exp(i\Lambda\varphi) \times \\
[\gamma(E^\star) + \xi \; \gamma(T-Z)] = \frac{Q_+}{2}\gamma(T+Z)\gamma(E^\star) + [m+\gamma(Q_T)]\,\gamma(E^\star) + \\
\xi \; \frac{Q_+}{2}\gamma(T+Z)\gamma(T-Z) + [m+\gamma(Q_T)]\,\xi \; \gamma(T-Z) + \\
\frac{Q_-}{2}\gamma(T-Z)\gamma(E^\star) = -\frac{Q_+}{2}\gamma(T+Z)\gamma(E^\star) + [m+\gamma(Q_T)]\,\gamma(E^\star) + \\
\xi \; \frac{Q_+}{2}\gamma(T+Z)\gamma(T-Z) + [m+\gamma(Q_T)]\,\xi \; \gamma(T-Z) - \frac{Q_-}{2}\gamma(E^\star)\gamma(T-Z)
\end{gathered} \tag{91}$$

which can be rewritten in the form (omitting the phase factor $\exp(i\Lambda\varphi)$)

$$\mathcal{N} = a\gamma(T+Z) + b + c\gamma(T-Z) \tag{92}$$

where

$$\begin{gathered}
a = -\frac{Q_+}{2}\gamma(E^\star) \\
b = [m+\gamma(Q_T)]\,\gamma(E^\star) + \xi \; \frac{Q_+}{2}\gamma(T+Z)\gamma(T-Z) \\
c = \xi \;\; [m+\gamma(Q_T)] - \frac{Q_-}{2}\gamma(E^\star)
\end{gathered} \tag{93}$$

The above arrangement has been performed according to the lowest orders of the various terms entering the matrix $\mathcal{N}$: thus, a is of (lowest) order 0, b is of (lowest) order 1 and c is of (lowest) order 2. In the following, by "order" of a term or of an expression we will mean their "lowest order" in the jet-like kinematics approximation. Thus, θ being a polar angle, $\cos(\theta)$ is of order zero and $\sin(\theta)$ is of order one ; $p_- = E - p_Z$ is of order two, etc.

The matrices a, b and c have the following properties

$$\begin{gathered}
a\gamma(T \pm Z) = \gamma(T \pm Z)a^{(1)} \quad \text{with} \quad a^{(1)} = -a \\
\gamma(T \pm Z)a = a^{(1)}\gamma(T \pm Z) \\
\gamma(T + Z)b = b^{(1)}\gamma(T + Z) \quad \text{with} \quad b^{(1)} = [\gamma(Q_T) - m]\,\gamma(E^\star) \\
\gamma(T - Z)b = b^{(2)}\gamma(T - Z) \quad \text{with} \quad b^{(2)} = b^{(1)} + \xi\,\frac{Q_+}{2}\gamma(T - Z)\gamma(T + Z) \\
b\gamma(T + Z) = \gamma(T + Z)b^{(2)} \\
b\gamma(T - Z) = \gamma(T - Z)b^{(1)} \\
\gamma(T \pm Z)b^{(1)} = b^{(3)}\gamma(T \pm Z) \quad \text{with} \quad b^{(3)} = [m + \gamma(Q_T)]\,\gamma(E^\star) \\
b^{(1)}\gamma(T \pm Z) = \gamma(T \pm Z)b^{(3)} \\
c\gamma(T \pm Z) = \gamma(T \pm Z)c^{(1)} \quad \text{with} \quad c^{(1)} = [m - \gamma(Q_T)]\,\xi + \frac{Q_-}{2}\gamma(E^\star) \\
\gamma(T \pm Z)c = c^{(1)}\gamma(T \pm Z)
\end{gathered} \tag{94}$$

Let us stress that in the expression (92) not only terms of various orders are well separated, but also matrices $\gamma(T + Z)$, 1 and $\gamma(T - Z)$ factorize, respectively, the zeroth order term, the first order term and the second order term. This suggests some simple underlying algebra.

Let us thus define matrices

$$\mathcal{N}_i = p_i + xq_i + x^2 r_i \tag{95}$$

where x is a real parameter and where

$$p_i = a_i\gamma(T + Z), \quad q_i = b_i, \quad r_i = c_i\gamma(T - Z) \tag{96}$$

the matrix factors a_i, b_i and c_i being defined as in (93) with appropriate labels. As will be seen below, introducing the parameter x provides a useful tool to track orders. The previous forms (92) of matrices $\mathcal{N}_i$ can be recovered by taking $x = 1$. Below we quote again the fundamental properties of matrices a_i, b_i and c_i that will be important in the following :

$$\begin{gathered}
\gamma(T \pm Z)a_i = a_i^{(1)}\gamma(T \pm Z), \quad a_i\gamma(T \pm Z) = \gamma(T \pm Z)a_i^{(1)} \\
\gamma(T + Z)b_i = b_i^{(1)}\gamma(T + Z), \quad b_i\gamma(T - Z) = \gamma(T - Z)b_i^{(1)}
\end{gathered} \tag{97}$$

$$
\begin{aligned}
b_i\gamma(T+Z) = \gamma(T+Z)b_i^{(2)}, &\quad \gamma(T-Z)b_i = b_i^{(2)}\gamma(T-Z) \\
\gamma(T\pm Z)c_i = c_i^{(1)}\gamma(T\pm Z), &\quad c_i\gamma(T\pm Z) = \gamma(T\pm Z)c_i^{(1)}
\end{aligned}
$$

Let us consider the product of two matrices $\mathcal{N}_1$ and $\mathcal{N}_2$. Since $\gamma(T\pm Z)^2 = 0$, we have

$$
\begin{gathered}
\mathcal{N}_2\mathcal{N}_1 = (a_2\gamma(T+Z) + b_2x + x^2c_2\gamma(T-Z))\left(\gamma(T+Z)a_1^{(1)} + xb_1 + \right. \\
\left. x^2\gamma(T-Z)c_1^{(1)}\right) = xa_2\gamma(T+Z)b_1 + x^2a_2\gamma(T+Z)\gamma(T-Z)c_1^{(1)} + \\
xb_2\gamma(T+Z)a_1^{(1)} + x^2b_2b_1 + x^3b_2\gamma(T-Z)c_1^{(1)} + \\
x^2c_2\gamma(T-Z)\gamma(T+Z)a_1^{(1)} + x^3c_2\gamma(T-Z)b_1
\end{gathered} \tag{98}
$$

Using the above-mentioned properties (97) of matrices a, b and c, it is possible to rewrite this product in the same form as (21) :

$$
\mathcal{N}_2\mathcal{N}_1 = x\left(A_2\gamma(T+Z) + xB_2 + x^2C_2\gamma(T-Z)\right) \tag{99}
$$

with

$$
\begin{gathered}
A_2 = b_2a_1 + a_2b_1^{(1)} \\
B_2 = b_2b_1 + c_2a_1^{(1)}\gamma(T-Z)\gamma(T+Z) + a_2c_1^{(1)}\gamma(T+Z)\gamma(T-Z) \\
C_2 = c_2b_1^{(2)} + b_2c_1
\end{gathered} \tag{100}
$$

As is revealed by the overall factor x, the multiplication of the two matrices increases orders by one unit. We see also that the above generic form provides again a clean separation of terms having different orders. Moreover, the hierarchy of orders is preserved : A_2 is now of first order, B_2 is of second order and C_2 of third order.

The new matrices A_2, B_2 and C_2 have the following properties :

$$
\begin{aligned}
A_2\gamma(T+Z) = \gamma(T+Z)A_2^{(1)}, &\quad \gamma(T-Z)A_2 = A_2^{(1)}\gamma(T-Z) \\
A_2\gamma(T-Z) = \gamma(T-Z)A_2^{(2)}, &\quad \gamma(T+Z)A_2 = A_2^{(2)}\gamma(T+Z) \\
A_2^{(1)}\gamma(T+Z) = \gamma(T+Z)A_2^{(3)}, &\quad \gamma(T-Z)A_2^{(1)} = A_2^{(3)}\gamma(T-Z)
\end{aligned} \tag{101}
$$

with

$$
A_2^{(1)} = b_2^{(2)}a_1^{(1)} + a_2^{(1)}b_1^{(3)},\ A_2^{(2)} = b_2^{(1)}a_1^{(1)} + a_2^{(1)}b_1^{(3)},\ A_2^{(3)} = b_2^{(3)}a_1 + a_2b_1^{(1)} \tag{102}
$$

$$
\begin{aligned}
\gamma(T+Z)B_2 = B_2^{(1)}\gamma(T+Z), &\quad B_2\gamma(T-Z) = \gamma(T-Z)B_2^{(1)} \\
\gamma(T-Z)B_2 = B_2^{(2)}\gamma(T-Z), &\quad B_2\gamma(T+Z) = \gamma(T+Z)B_2^{(2)} \\
B_2^{(1)}\gamma(T-Z) = \gamma(T-Z)B_2^{(3)}, &\quad \gamma(T+Z)B_2^{(1)} = B_2^{(3)}\gamma(T+Z)
\end{aligned} \tag{103}
$$

with

$$
\begin{aligned}
B_2^{(1)} &= b_2^{(1)} b_1^{(1)} + \gamma(T+Z) c_2 a_1^{(1)} \gamma(T-Z) + \gamma(T-Z) c_2 a_1^{(1)} \gamma(T+Z) \\
B_2^{(2)} &= b_2^{(2)} b_1^{(2)} + \gamma(T-Z) a_2 c_1^{(1)} \gamma(T+Z) + \gamma(T+Z) a_2 c_1^{(1)} \gamma(T-Z) \\
B_2^{(3)} &= b_2^{(3)} b_1^{(3)} + \gamma(T+Z) c_2^{(1)} a_1 \gamma(T-Z) + \gamma(T-Z) c_2^{(1)} a_1 \gamma(T+Z)
\end{aligned} \tag{104}
$$

$$
\begin{aligned}
C_2 \gamma(T-Z) = \gamma(T-Z) C_2^{(1)}, &\quad \gamma(T+Z) C_2 = C_2^{(1)} \gamma(T+Z) \\
C_2 \gamma(T+Z) = \gamma(T+Z) C_2^{(2)}, &\quad \gamma(T-Z) C_2 = C_2^{(2)} \gamma(T-Z)
\end{aligned} \tag{105}
$$

with

$$
C_2^{(1)} = c_2^{(1)} b_1 + b_2^{(1)} c_1^{(1)}, \quad C_2^{(2)} = c_2^{(1)} b_1^{(3)} + b_2^{(2)} c_1^{(1)} \tag{106}
$$

Thus, it is seen that due to the structure of the matrices A, B or C, which are constructed from $\gamma(X)$, $\gamma(Y)$, and from the products $\gamma(T+Z)\gamma(T-Z)$ and $\gamma(T-Z)\gamma(T+Z)$, any product like $\gamma(T \pm Z)\mathcal{X}$ or $\mathcal{X}\gamma(T \pm Z)$, $\mathcal{X}$ being A, B or C, gives $\mathcal{X}'\gamma(T \pm Z)$ and $\gamma(T \pm Z)\mathcal{X}''$ respectively, where $\mathcal{X}'$ and $\mathcal{X}''$ have a similar structure as that of $\mathcal{X}$. As a consequence, the product of any number of matrices such as $\mathcal{N}$ has the same structure as that of $\mathcal{N}$. To prove this explicitly, we may proceed by recurrence. Let N_n be the matrix $N_n = A_n\gamma(T+Z) + B_n + C_n\gamma(T-Z)$ where the matrices A_n, B_n and C_n have the general properties described by formulas (101), (103) and (105), and $\mathcal{N}_{n+1}$ a matrix like (90), (92). We have

$$
\begin{gathered}
N_{n+1} = \mathcal{N}_{n+1} N_n = \left[a_{n+1}\gamma(T+Z) + b_{n+1} + c_{n+1}\gamma(T-Z)\right]\left[A_n\gamma(T+Z) + \right. \\
\left. + B_n + C_n\gamma(T-Z)\right] = a_{n+1}\gamma(T+Z)A_n\gamma(T+Z) + a_{n+1}\gamma(T+Z)B_n + \\
a_{n+1}\gamma(T+Z)C_n\gamma(T-Z) + b_{n+1}A_n\gamma(T+Z) + b_{n+1}B_n + \\
b_{n+1}C_n\gamma(T-Z) + c_{n+1}\gamma(T-Z)A_n\gamma(T+Z) + c_{n+1}\gamma(T-Z)B_n + \\
c_{n+1}\gamma(T-Z)C_n\gamma(T-Z) = a_{n+1}A_n^{(1)}\gamma(T+Z)^2 + a_{n+1}B_n^{(1)}\gamma(T+Z) + \\
\gamma(T+Z)a_{n+1}^{(1)}C_n\gamma(T-Z) + b_{n+1}A_n\gamma(T+Z) + b_{n+1}B_n + b_{n+1}C_n\gamma(T-Z) + \\
\gamma(T-Z)c_{n+1}^{(1)}A_n\gamma(T+Z) + c_{n+1}B_n^{(2)}\gamma(T-Z) + c_{n+1}\gamma(T-Z)^2 C_n^{(1)}
\end{gathered} \tag{107}
$$

Taking care of hierarchy of orders, this product may be written in the form

$$
N_{n+1} = A_{n+1}\gamma(T+Z) + B_{n+1} + C_{n+1}\gamma(T-Z) \tag{108}
$$

where

$$
\begin{gathered}
A_{n+1} = b_{n+1}A_n + a_{n+1}B_n^{(1)} \\
B_{n+1} = b_{n+1}B_n + \gamma(T-Z)c_{n+1}^{(1)}A_n\gamma(T+Z) + \gamma(T+Z)a_{n+1}^{(1)}C_n\gamma(T-Z) \\
C_{n+1} = c_{n+1}B_n^{(2)} + b_{n+1}C_n
\end{gathered} \tag{109}
$$

In fact, the preceding calculation shows that it is sufficient to prove that the relations

$$\begin{aligned} A_n\gamma(T+Z) &= \gamma(T+Z)A_n^{(1)} \\ \gamma(T+Z)B_n &= B_n^{(1)}\gamma(T+Z) \\ \gamma(T-Z)B_n &= B_n^{(2)}\gamma(T-Z) \\ C_n\gamma(T-Z) &= \gamma(T-Z)C_n^{(1)} \end{aligned} \tag{110}$$

hold true for any matrix $\mathcal{N}$, i.e. for all n. They are obviously true for $n = 1$. Assuming they are true for some n, i.e. for the matrices A_n, B_n and C_n, then, from the recurrence relations (109) we have

$$\begin{gathered} A_{n+1}\gamma(T+Z) = a_{n+1}B_n^{(1)}\gamma(T+Z) + b_{n+1}A_n\gamma(T+Z) = \\ \gamma(T+Z)a_{n+1}^{(1)}B_n + \gamma(T+Z)b_{n+1}^{(2)}A_n^{(1)} = \\ \gamma(T+Z)A_{n+1}^{(1)} \quad \text{with} \quad A_{n+1}^{(1)} = a_{n+1}^{(1)}B_n + b_{n+1}^{(2)}A_n^{(1)} \end{gathered} \tag{111}$$

$$\begin{gathered} \gamma(T+Z)B_{n+1} = \gamma(T+Z)b_{n+1}B_n + \gamma(T+Z)\gamma(T-Z)c_{n+1}^{(1)}A_n\gamma(T+Z) = \\ b_{n+1}^{(1)}B_n^{(1)}\gamma(T+Z) + \gamma(T+Z)\gamma(T-Z)c_{n+1}^{(1)}A_n\gamma(T+Z) = \\ B_{n+1}^{(1)}\gamma(T+Z) \quad \text{with} \quad B_{n+1}^{(1)} = b_{n+1}^{(1)}B_n^{(1)} + \gamma(T+Z)\gamma(T-Z)c_{n+1}^{(1)}A_n \end{gathered} \tag{112}$$

$$\begin{gathered} \gamma(T-Z)B_{n+1} = \gamma(T-Z)b_{n+1}B_n + \gamma(T-Z)\gamma(T+Z)a_{n+1}^{(1)}C_n\gamma(T-Z) = \\ b_{n+1}^{(2)}B_n^{(2)}\gamma(T-Z) + \gamma(T-Z)\gamma(T+Z)a_{n+1}^{(1)}C_n\gamma(T-Z) = \\ B_{n+1}^{(2)}\gamma(T-Z) \quad \text{with} \quad B_{n+1}^{(2)} = b_{n+1}^{(2)}B_n^{(2)} + \gamma(T-Z)\gamma(T+Z)a_{n+1}^{(1)}C_n \end{gathered} \tag{113}$$

$$\begin{gathered} C_{n+1}\gamma(T-Z) = c_{n+1}B_n^{(2)}\gamma(T-Z) + b_{n+1}C_n\gamma(T-Z) = \\ \gamma(T-Z)c_{n+1}^{(1)}B_n + \gamma(T-Z)b_{n+1}^{(1)}C_n^{(1)} = \gamma(T-Z)C_{n+1}^{(1)} \\ \text{with} \quad C_{n+1}^{(1)} = c_{n+1}^{(1)}B_n + b_{n+1}^{(1)}C_n^{(1)} \end{gathered} \tag{114}$$

Thus, properties (110) also hold for the matrices A_{n+1}, B_{n+1} and C_{n+1} ; Q.E.D.

Now, if all properties given in (101), (103) and (105) hold for the matrices A_n, B_n and C_n, then we have in addition

$$\begin{gathered} \gamma(T-Z)A_{n+1} = \gamma(T-Z)a_{n+1}B_n^{(1)} + \gamma(T-Z)b_{n+1}A_n = \\ a_{n+1}^{(1)}B_n\gamma(T-Z) + b_2^{(2)}A_n^{(1)}\gamma(T-Z) = A_{n+1}^{(1)}\gamma(T-Z) \end{gathered} \tag{115}$$

$$
\begin{aligned}
A_{n+1}\gamma(T-Z) = a_{n+1}B_n^{(1)}\gamma(T-Z) + b_{n+1}A_n\gamma(T-Z) = \\
\gamma(T-Z)a_{n+1}^{(1)}B_n^{(3)} + \gamma(T-Z)b_{n+1}^{(1)}A_n^{(2)} = \gamma(T-Z)A_{n+1}^{(2)} \\
\text{with} \quad A_{n+1}^{(2)} = a_{n+1}^{(1)}B_n^{(3)} + b_{n+1}^{(1)}A_n^{(2)}
\end{aligned} \tag{116}
$$

$$
\begin{aligned}
\gamma(T+Z)A_{n+1} = \gamma(T+Z)a_{n+1}B_n^{(1)} + \gamma(T+Z)b_{n+1}A_n = \\
a_{n+1}^{(1)}B_n^{(3)}\gamma(T+Z) + b_{n+1}^{(1)}A_n^{(2)}\gamma(T+Z) = A_{n+1}^{(2)}\gamma(T+Z)
\end{aligned} \tag{117}
$$

$$
\begin{aligned}
A_{n+1}^{(1)}\gamma(T+Z) = a_{n+1}^{(1)}B_n\gamma(T+Z) + b_{n+1}^{(2)}A_n^{(1)}\gamma(T+Z) = \\
\gamma(T+Z)a_{n+1}B_n^{(2)} + \gamma(T+Z)b_{n+1}^{(3)}A_n^{(3)} = \gamma(T+Z)A_{n+1}^{(3)} \\
\text{with} \quad A^{(3)} = a_{n+1}B_n^{(2)} + b_{n+1}^{(3)}A_n^{(3)}
\end{aligned} \tag{118}
$$

$$
\begin{aligned}
\gamma(T-Z)A_{n+1}^{(1)} = \gamma(T-Z)a_{n+1}^{(1)}B_n + \gamma(T-Z)b_{n+1}^{(2)}A_n^{(1)} = \\
a_{n+1}B_n^{(2)}\gamma(T-Z) + b_{n+1}^{(3)}A_n^{(3)}\gamma(T-Z) = A_{n+1}^{(3)}\gamma(T+Z)
\end{aligned} \tag{119}
$$

$$
\begin{aligned}
\gamma(T+Z)B_{n+1} = \gamma(T+Z)b_{n+1}B_n + \gamma(T+Z)\gamma(T-Z)c_{n+1}^{(1)}A_n\gamma(T+Z) = \\
b_{n+1}^{(1)}B_n^{(1)}\gamma(T+Z) + \gamma(T+Z)c_{n+1}A_n^{(1)}\gamma(T-Z)\gamma(T+Z) \\
= \gamma(T+Z)B_{n+1}^{(1)}
\end{aligned} \tag{120}
$$

with

$$
\begin{aligned}
B_{n+1}^{(1)} = b_{n+1}^{(1)}B_n^{(1)} + \gamma(T+Z)c_{n+1}A_n^{(1)}\gamma(T-Z) + \\
\gamma(T-Z)c_{n+1}A_n^{(1)}\gamma(T+Z)
\end{aligned} \tag{121}
$$

$$
\begin{aligned}
B_{n+1}\gamma(T-Z) = b_{n+1}B_n\gamma(T-Z) + \gamma(T-Z)c_{n+1}^{(1)}A_n\gamma(T+Z)\gamma(T-Z) = \\
\gamma(T-Z)b_{n+1}^{(1)}B_n^{(1)} + \gamma(T-Z)\gamma(T+Z)c_{n+1}A_n^{(1)}\gamma(T-Z) = \\
\gamma(T-Z)B_{n+1}^{(1)}
\end{aligned} \tag{122}
$$

$$
\begin{aligned}
\gamma(T-Z)B_{n+1} = \gamma(T-Z)b_{n+1}B_n + \gamma(T-Z)\gamma(T+Z)a_{n+1}^{(1)}C_n\gamma(T-Z) = \\
b_{n+1}^{(2)}B_n^{(2)}\gamma(T-Z) + \gamma(T-Z)a_{n+1}C_n^{(1)}\gamma(T+Z)\gamma(T-Z) = \\
B_{n+1}^{(2)}\gamma(T-Z)
\end{aligned} \tag{123}
$$

with

$$
\begin{aligned}
B_{n+1}^{(2)} = b_{n+1}^{(2)}B_n^{(2)} + \gamma(T-Z)a_{n+1}C_n^{(1)}\gamma(T+Z) + \\
\gamma(T+Z)a_{n+1}C_n^{(1)}\gamma(T-Z)
\end{aligned} \tag{124}
$$

$$B_{n+1}\gamma(T+Z) = b_{n+1}B_n\gamma(T+Z) + \gamma(T+Z)a^{(1)}_{n+1}C_n\gamma(T-Z)\gamma(T+Z) = \\ \gamma(T+Z)b^{(2)}_{n+1}B^{(2)}_n + \gamma(T+Z)a_{n+1}C^{(1)}_n\gamma(T-Z)\gamma(T+Z) = \\ \gamma(T+Z)B^{(2)}_{n+1} \tag{125}$$

$$B^{(1)}_{n+1}\gamma(T-Z) = b^{(1)}_{n+1}B^{(1)}_n\gamma(T-Z) + \gamma(T-Z)\gamma(T+Z)c_{n+1}A^{(1)}_n\gamma(T-Z) = \\ \gamma(T-Z)b^{(3)}_{n+1}B^{(3)}_n + \gamma(T-Z)\gamma(T+Z)c^{(1)}_{n+1}A^{(3)}_n\gamma(T-Z) = \\ \gamma(T-Z)B^{(3)}_{n+1} \tag{126}$$

with

$$B^{(3)}_{n+1} = b^{(3)}_{n+1}B^{(3)}_n + \gamma(T+Z)c^{(1)}_{n+1}A^{(3)}_n\gamma(T-Z) + \\ \gamma(T-Z)c^{(1)}_{n+1}A^{(3)}_n\gamma(T+Z) \tag{127}$$

$$\gamma(T+Z)B^{(1)}_{n+1} = \gamma(T+Z)b^{(1)}_{n+1}B^{(1)}_n + \gamma(T+Z)\gamma(T-Z)c_{n+1}A^{(1)}_n\gamma(T+Z) = \\ b^{(3)}_{n+1}B^{(3)}_n\gamma(T+Z) + \gamma(T+Z)c^{(1)}_{n+1}A^{(3)}_n\gamma(T-Z)\gamma(T+Z) = \quad (128) \\ B^{(3)}_{n+1}\gamma(T+Z) \quad (129)$$

$$\gamma(T+Z)C_{n+1} = \gamma(T+Z)c_{n+1}B^{(2)}_n + \gamma(T+Z)b_{n+1}C_n = \\ c^{(1)}_{n+1}B_n\gamma(T+Z) + b^{(1)}_{n+1}C^{(1)}_{n+1}\gamma(T+Z) = C^{(1)}_{n+1}\gamma(T+Z) \tag{130}$$

Without prejudging the practical usefulness of the above development, at least the latter has the advantage of providing us with a way to track orders. To show this, let us consider the product

$$N_n = \mathcal{N}_n\mathcal{N}_{n-1}\cdots\mathcal{N}_1 = \left(p_n + xq_n + x^2r_n\right)\cdots\left(p_1 + xq_1 + x^2r_1\right) \tag{131}$$

From the preceding development, making the substitutions $b_i \to xb_i$ and $c_i \to x^2c_i$, we find

$$N_n = x^{n-1}\left(P_n + xQ_n + x^2R_n\right) \tag{132}$$

with

$$P_n = A_n\gamma(T+Z), \qquad Q_n = B_n, \qquad R_n = C_n\gamma(T-Z) \tag{133}$$

where the matrices A_n, B_n and C_n satisfy the recurrence relations (109).

It is worth noticing that in N_n the overall power factor x^{n-1} fixes the lowest order (i.e. $n-1$) of the terms entering that matrix. Moreover, terms of different orders are separated still in a clean way, and hierarchy of orders is preserved : A_n is of order $n-1$, B_n is of order n and C_n is of order $n+1$. Notice also that in some sense matrices $\gamma(T+Z)$ and $\gamma(T-Z)$ play the role of projectors with respect to orders.

The same kind of treatment can be applied as well to a matrix like

$$\mathcal{N}' = \gamma(\epsilon^\star)[m + \gamma(Q')] \tag{134}$$

and products of such matrices, so that the matrix $\mathcal{M}$ in (86) can be written in the form

$$\begin{aligned}\mathcal{M} = \left[\gamma(T+Z)\bar{A}' + \bar{B}' + \gamma(T-Z)\bar{C}'\right]\gamma(T-Z)\times \\ [A\gamma(T+Z) + B + C\gamma(T-Z)]\end{aligned} \tag{135}$$

i.e.

$$\mathcal{M} = \left[\gamma(T+Z)\bar{A}' + \bar{B}'\right]\gamma(T-Z)\left[A\gamma(T+Z) + B\right] \tag{136}$$

$$\begin{aligned}\mathcal{M} = 4\bar{A}'^{(1)}\gamma(T+Z)A^{(1)} + \bar{A}'^{(1)}\gamma(T+Z)\gamma(T-Z)B + \\ \bar{B}'\gamma(T-Z)\gamma(T+Z)A^{(1)} + \bar{B}'\gamma(T-Z)B\end{aligned} \tag{137}$$

It is interesting to notice that taking the light-like 4-vector $T-Z$ as a polarization 4-vector of the virtual photon simply kills the higher order tems contained in factors $\gamma(T-Z)C$. Thus, the matrix (137) comprises a first piece $\bar{A}\gamma(T+Z)A^{(1)}$ the order of which is $p-2+N-p-1 = n_\gamma - 2$. This is the lowest order term of the matrix. There are then two terms $\bar{A}\gamma(T+Z)\gamma(T-Z)B$ and $\bar{B}\gamma(T-Z)\gamma(T+Z)A^{(1)}$ of order $n_\gamma - 1$. The last term $\bar{B}\gamma(T-Z)B$ is of order n_γ. We already know that the amplitude J in (86) should be of order n_γ. So, we may expect that when the matrix $\mathcal{M}$ is sandwiched in the scalar product J, the lowest order term of (137) picks up terms of overall order 2 coming from the external spinors, while the terms of order $n_\gamma - 1$ and the last term of order n_γ pick up terms of overall order 1 and 0 respectively. The least we can say is that we should be very careful with respect to orders when we compute the amplitude in the framework of jet-like kinematics approximation, because all terms conspire to produce an amplitude of final order n_γ. Hopefully, as will be seen in the next section, to find out the contribution due to the first term of (137), it is sufficient to consider the development of external spinors up to the order one only, because this term picks up the terms of order one in each of the two external spinors !

To end up this section, let us make the following comments.

a) Why did we decide to consider the effect of the product of matrices $[m+\gamma(Q)]$ and $\gamma(\epsilon^\star)$ instead of considering the effect of each matrix separately ? The former matrix involves terms of order zero, one and two with the same structure as in (92).

In contrast with this, the second matrix is a combination of terms of order zero and one only, and has a different matrix structure : the zero order term is $\gamma(E^\star)$ and the first order term has the factor $\gamma(T-Z)$. To obtain for this matrix a structure similar to (92), it would have been necessary to multiply it by some first order term, say the lepton mass. But, anyway, we know that second order terms should not be discarded a priori. So instead of using some complicated trick, we found it more convenient to consider the product of these two matrices, which takes on the structure (92) that leads to the nice algebra described above with a clean separation of orders.

b) The introduction of the parameter x in (95) has its origin in the observation that a boost of rapidity χ along the $Z-$axis transforms $\gamma(T+Z)$, $\gamma(E^\star)$ and $\gamma(T-Z)$ into, respectively, $\exp(\chi)\gamma(T+Z)$, $\gamma(E^\star)$ and $\exp(-\chi)\gamma(T-Z)$, and a matrix like (92) is transformed into $\exp(\chi)\left[a\gamma(T+Z)+\exp(-\chi)b+\exp(-2\chi)c\gamma(T-Z)\right]$. So, it appears that a boost along the $Z-$axis distinguishes between the orders. This simple observation led us to the idea of introducing the parameter x.

5.3 Further possible treatment of the problem

Taking into account the first relation in (8), it is convenient to rewrite the spinors U_1 associated with the incoming lepton in the form

$$U_1^\lambda = \sqrt{m}\left[\exp(\frac{\chi_1}{2})\gamma(T+Z)+\exp(-\frac{\chi_1}{2})\gamma(T-Z)\right]U_0^\lambda/2 \qquad \text{or}$$
$$U_1^\lambda = \sqrt{\frac{E_1}{2}}\left[\gamma(T+Z)\alpha+\gamma(T-Z)\beta\right] \tag{138}$$

where

$$\alpha = \sqrt{\frac{2m}{E_1}}\exp(\frac{\chi_1}{2})U_0^\lambda = \frac{1}{2}\left[\sqrt{1+m/E_1}+\sqrt{1-m/E_1}\right]U_0^\lambda \tag{139}$$

and

$$\beta = \sqrt{\frac{2m}{E_1}}\exp(-\frac{\chi_1}{2})U_0^\lambda = \frac{1}{2}\left[\sqrt{1+m/E_1}-\sqrt{1-m/E_1}\right]U_0^\lambda \tag{140}$$

Notice that in the jet-like kinematics approximation, β is found less than α by one order.

Now, let us see what is the result of the application of an elementary matrix $\mathcal{N}$ on a spinor $\Psi=\gamma(T+Z)\alpha+\gamma(T-Z)\beta$. We get

$$\begin{aligned}\mathcal{N}\Psi = \left[a\gamma(T+Z)+b+c\gamma(T-Z)\right]\left[\gamma(T+Z)\alpha+\gamma(T-Z)\beta\right] = \\ a\gamma(T+Z)\gamma(T-Z)\beta+b\left[\gamma(T+Z)\alpha+\gamma(T-Z)\beta\right]+ \\ c\gamma(T-Z)\gamma(T+Z)\alpha\end{aligned} \tag{141}$$

But, from (97), we have

$$\begin{aligned}
a\gamma(T+Z)\gamma(T-Z) = \gamma(T+Z)a^{(1)}\gamma(T-Z) \\
b\gamma(T-Z) = \gamma(T-Z)b^{(1)}, \quad b\gamma(T+Z) = \gamma(T+Z)b^{(2)} \\
c\gamma(T-Z)\gamma(T+Z) = \gamma(T-Z)c^{(1)}\gamma(T+Z)
\end{aligned} \tag{142}$$

Then,

$$\begin{aligned}
\mathcal{N}\Psi = \gamma(T+Z)\ a^{(1)}\ \gamma(T-Z)\ \beta + \gamma(T+Z)\ b^{(2)}\ \alpha + \gamma(T-Z)\ b^{(1)}\ \beta + \\
\gamma(T-Z)\ c^{(1)}\ \gamma(T+Z)\ \alpha \ = \ \gamma(T+Z)\ \left[b^{(2)}\ \alpha + a^{(1)}\ \gamma(T-Z)\ \beta\right] + \\
\gamma(T-Z)\ \left[b^{(1)}\ \beta + c^{(1)}\ \gamma(T+Z)\alpha\ \right]
\end{aligned} \tag{143}$$

which is of the form

$$\Psi' = \gamma(T+Z)\alpha' + \gamma(T-Z)\beta' \tag{144}$$

provided we set

$$\alpha' = b^{(2)}\alpha + a^{(1)}\gamma(T-Z)\beta \qquad \text{and} \qquad \beta' = b^{(1)}\beta + c^{(1)}\gamma(T+Z)\alpha \tag{145}$$

These two last relations may be conveniently recast in matrix form

$$\begin{pmatrix} \alpha' \\ \beta' \end{pmatrix} = \mathcal{Y} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \tag{146}$$

with the 8X8 matrix $\mathcal{Y}$ given by

$$\mathcal{Y} = \begin{pmatrix} b^{(2)} & a^{(1)}\gamma(T-Z) \\ c^{(1)}\gamma(T+Z) & b^{(1)} \end{pmatrix} \tag{147}$$

Thus, the effect of the matrix $\mathcal{N}$ can be understood on a 8-dimensional space as follows

$$\Psi' = \mathcal{N}\Psi \equiv \begin{pmatrix} \gamma(T+Z) & 0 \\ 0 & \gamma(T-Z) \end{pmatrix} \mathcal{Y} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \tag{148}$$

It is then straightforward to find the effect of several matrices like $\mathcal{N}$ on Ψ

$$\mathcal{N}_p\mathcal{N}_{p-1}\cdots\mathcal{N}_1\Psi \equiv \begin{pmatrix} \gamma(T+Z) & 0 \\ 0 & \gamma(T-Z) \end{pmatrix} \mathcal{Y} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \tag{149}$$

with

$$\mathcal{Y} = \mathcal{Y}_p \mathcal{Y}_{p-1} \cdots \mathcal{Y}_1 \tag{150}$$

each matrix $\mathcal{Y}_k$ being given by an expression like (147).

The main interest of such a presentation is that here again orders are well separated. Thus in (76), the matrix elements at the right top are of order zero ; the diagonal elements are of order one ; the matrix elements at the left bottom are of second order. Acting on the vector of components α and β, a matrix $\mathcal{Y}_i$ raises the order of these components by one unit, but preserves the hierarchy of order between the resulting components α' and β' : β' remains one order less than α'. Thus, this formalism provides a nice description of the propagation of orders along the string of matrices in the amplitude (86).

Taking into account the results of the preceding section, the part of the matrix (86) that is at the right of $\gamma(T-Z)$ can then be represented by

$$\mathcal{N}_r = \begin{pmatrix} \gamma(T+Z) & 0 \\ 0 & \gamma(T-Z) \end{pmatrix} \mathcal{Y}_r \tag{151}$$

with

$$\mathcal{Y}_r = \mathcal{Y}_{p-1} \mathcal{Y}_{p-2} \cdots \mathcal{Y}_1 = \begin{pmatrix} B_r^{(2)} & A_r^{(1)} \gamma(T-Z) \\ C_r^{(1)} \gamma(T+Z) & B_r^{(1)} \end{pmatrix} \tag{152}$$

Obviously, the same treatment can be applied as well to a matrix like

$$\bar{\mathcal{N}} = \gamma(\epsilon^\star)[m + \gamma(Q')] \tag{153}$$

and products of such matrices (notice the relation $\bar{\mathcal{N}}_\Lambda(Q) = -\gamma_0 \mathcal{N}^\dagger_{-\Lambda}(Q)\gamma_0$). Thus, the part of the matrix (86) that is at the left of $\gamma(T-Z)$ can be represented by

$$\mathcal{N}_\ell = \mathcal{Y}_\ell \begin{pmatrix} \gamma(T+Z) & 0 \\ 0 & \gamma(T-Z) \end{pmatrix} \tag{154}$$

with

$$\mathcal{Y}_\ell = \mathcal{Y}_N \mathcal{Y}_{N-1} \cdots \mathcal{Y}_{p+1} = \begin{pmatrix} \bar{B}_\ell^{(2)} & \gamma(T+Z)\bar{C}_\ell^{(1)} \\ \gamma(T-Z)\bar{A}_\ell^{(1)} & \bar{B}_\ell^{(1)} \end{pmatrix} \tag{155}$$

In appendix 6.1 we show that the spinors U_3 of the outgoing lepton can also be written in a form analogous to the last expression of U_1 in (138). Thus, we arrive at the following general expression for the amplitude in (86) (taking (137) into account)

$$J = \bar{U}_3 \mathcal{M} U_1 = \frac{1}{2}\sqrt{E_3E_1}\exp(i\lambda_3\phi_3/2)\left[\bar{\alpha}_3\gamma(T+Z)+\bar{\beta}_3\gamma(T-Z)\right]\times$$
$$\Big[4\bar{A}_\ell^{(1)}\gamma(T+Z)A_r^{(1)}+\bar{A}_\ell^{(1)}\gamma(T+Z)\gamma(T-Z)B_r+ \tag{156}$$
$$\bar{B}_\ell\gamma(T-Z)\gamma(T+Z)A_r^{(1)}+\bar{B}_\ell\gamma(T-Z)B_r\Big]\left[\gamma(T+Z)\alpha_1+\gamma(T-Z)\beta_1\right]$$

i.e.

$$J = \frac{1}{2}\sqrt{E_3E_1}\exp(i\lambda_3\phi_3/2)\Big[16\ \bar{\beta}_3\bar{A}_\ell\gamma(T-Z)A_r\beta_1+$$
$$4\ \bar{\beta}_3\bar{A}_\ell\gamma(T-Z)B_r\gamma(T+Z)\alpha_1+4\ \bar{\alpha}_3\gamma(T+Z)\bar{B}_\ell\gamma(T-Z)A_r\beta_1+ \tag{157}$$
$$4\ \bar{\alpha}_3\bar{B}_\ell^{(2)}\gamma(T+Z)B_r^{(2)}\alpha_1\Big]$$

From this last result we may draw the following important conclusions. First, it is manifest that the expression in brackets in (157) has lowest order n_γ. As expected, the lowest order term in the matrix (137) selects the one order parts β_3 and β_1 of external spinors to give a final expression of order n_γ. In addition, the ratio J/E_1 is, as expected, independent on the incident energy E_1.

Next, we arrive at a formidable nontrivial conclusion. Up to now, we made exact calculations and (137) is an exact expression for the generic form of impact factors. But if we go to jet-like kinematics conditions, this expression shows us that to obtain the impact factor at lowest order, it is sufficient to take in matrices $\mathcal{N}_i$ each term equal to its lowest order expression. However, we should take each term into account. In particular, we cannot generally discard second order terms such as those proportionnal to Q_- because they do participate to the full amplitude.

Finally, let us remark that if the virtual photon is at the first place in the Feynman diagram (reading the latter from right to left), we should obviously take $A_r = 0,\ B_r = 1,\ C_r = 0$. If the virtual photon is at the last place, we should take $\bar{A}_l = 0,\ \bar{B}_l = 1,\ \bar{C}_l = 0$.

5.4 Particular cases

Simple general properties of impact factors may be established for some particular cases : the case where all final particles are emitted in the strict forward direction ; the case where one final photon takes away the whole energy of the incoming lepton.

1°) The emission in the strict forward direction : this has been already considered in section 4.

2°) One photon's x is close to one.

Here, the x of a photon of 4-momentum k is the ratio k_+/p_{1+}, which at lowest order is the fraction of energy taken away by the photon.

Assume first this energetic photon ($k_+/p_{1+} \to 1$) being emitted at the jth place before the vertex of the virtual photon. Then in every matrix $\mathcal{N}_i$ in (86) with $i < j$ we may take the 4-momenta of the involved (soft) photons equal to zero, and we then get

$$\mathcal{N}_{j-1}\mathcal{N}_{j-2}\cdots\mathcal{N}_1 U_1^\lambda = [m+\gamma(p_1)]\,\gamma(\epsilon_{j-1}^\star)\cdots[m+\gamma(p_1)]\,\gamma(\epsilon_1^\star)U_1^\lambda = \left(2p_1.\epsilon_{j-1}^\star\right)\cdots(2p_1.\epsilon_1^\star)\,U_1^\lambda \tag{158}$$

Next, we turn to the effect of subsequent matrices. We said in the previous section that it is possible to take each term equal to its lowest order expression. Thus, in matrices $\mathcal{N}_m$ with $m \geq j$ we may take $Q_+ \equiv (p_1 - k_j)_+ = 0$ (the 4-momenta of the other emitted soft-photons are again set to zero). Thus, those matrices may be taken in the form

$$\mathcal{N}_m \equiv \left[m + \frac{Q_-}{2}\gamma(T-Z) + \gamma(Q_T)\right]\gamma(\epsilon_m^\star) \tag{159}$$

for propagators taking place before the vertex of the virtual photon, or

$$\mathcal{N}_n \equiv \gamma(\epsilon_n^\star)\left[m + \frac{Q_-}{2}\gamma(T-Z) + \gamma(Q_T)\right] \tag{160}$$

for propagators involved after that vertex.

For both types of matrices the application of the matrix $\gamma(T-Z)$ kills the second order terms $\propto Q_-$'s and also the term $\propto \xi$'s in matrices $\gamma(\epsilon^\star)$. We thus get

$$\begin{gathered}\bar{\mathcal{N}}_N\cdots\bar{\mathcal{N}}_{p+1}\gamma(T-Z)\mathcal{N}_{p-1}\cdots\mathcal{N}_j U_1^\lambda \propto \\ \gamma(E_N^\star)\,[m+\gamma(Q_{TN})]\cdots\gamma(E_{p+1}^\star)\,[m+\gamma(Q_{Tp+1})]\,\gamma(T-Z)\times \\ [m+\gamma(Q_{Tp-1})]\,\gamma(E_{p-1}^\star)\cdots[m+\gamma(Q_{Tj})]\,\gamma(E_j^\star)U_1^\lambda\end{gathered} \tag{161}$$

But (see Eq. (C.3) in appendix 6.3)

$$\gamma(E_j^\star)U_1^\lambda = -2\ \lambda\ \sqrt{2}\ \delta_{\Lambda_j,2\lambda}\ V_1^{-\lambda} \tag{162}$$

Because of the factor $\delta_{\Lambda_j,2\lambda}$, we conclude that in such a case, the initial lepton "transmits" its helicity to that jth energetic photon. We obtain the same conclusion if the energetic photon is emitted after the vertex of the virtual photon. Thus, this property holds true for the whole impact factor itself (where all graphs are taken into account).

5.5 Helicity properties of matrix factors A, B and C

From the general relations (101-105), we obtain

$$\gamma(T+Z)\gamma(T-Z)A = \gamma(T+Z)A^{(1)}\gamma(T-Z) = A\gamma(T+Z)\gamma(T-Z) \tag{163}$$

thus, A commutes with $\gamma(T+Z)\gamma(T-Z)$ or, equivalently, with the boost operator $\gamma(Z)\gamma(T)/2$. We get the same conclusion for B and C since

$$\gamma(T-Z)\gamma(T+Z)B = \gamma(T-Z)B^{(1)}\gamma(T+Z) = B\gamma(T-Z)\gamma(T+Z) \tag{164}$$

and

$$C\gamma(T-Z)\gamma(T+Z) = \gamma(T-Z)C^{(1)}\gamma(T+Z) = \gamma(T-Z)\gamma(T+Z)C \tag{165}$$

As the helicity operator is given by $S_Z = \gamma_5\gamma(Z)\gamma(T)/2$, the helicity properties of A, B and C are then determined by their commutation rule with the chiral operator γ_5. From (93) it is seen that each of the elementary matrices a, b and c can be divided in two parts : a part proportional to the lepton mass m and a part that does not contain the lepton mass. Let us label the first one by a (o) for "odd" and the second one by a (e) for "even" :

$$a^o = 0, \quad a^e = -\frac{Q_+}{2}\gamma(E^\star) \tag{166}$$

$$b^o = m\gamma(E^\star), \quad b^e = \gamma(Q_T)\gamma(E^\star) + \xi\frac{Q_+}{2}\gamma(T+Z)\gamma(T-Z) \tag{167}$$

$$c^o = m\xi, \quad c^e = \xi\gamma(Q_T) - \frac{Q_-}{2}\gamma(E^\star) \tag{168}$$

Taking into account the fact that the matrices $\gamma(T \pm Z)$ are helicity conserving, and that the "transverse" matrices $\gamma(E^{(\pm)})$ are helicity flipping, it appears that

- the odd parts a^o and c^o are helicity conserving while the even parts a^e and c^e do not conserve helicity ; this is also true for $a^{(1)}$ and $c^{(1)}$;
- the odd part b^o does not conserve helicity whereas the even part b^e conserves helicity ; this is also true for $b^{(1)}$ and $b^{(2)}$.

These properties can be generalized by recurrence to any matrix factors A_n, B_n and C_n. It is always possible to divide these factors in a part involving odd powers of m (labelled by a (o)) and a part involving even powers of m (labelled by a (e)) :

$$\begin{aligned} A_n &= A_n^o + A_n^e \\ B_n &= B_n^o + B_n^e \\ C_n &= C_n^o + C_n^e \end{aligned} \tag{169}$$

Let us assume the above itemized properties to be true for the rank n. Then, from the recurrence relations (109) we have

$$\begin{gathered} A_{n+1}^o = b_{n+1}^o A_n^e + b_{n+1}^e A_n^o + a_{n+1}^e B_n^{(1)o} + a_{n+1}^o B_n^{(1)e} \\ A_{n+1}^e = b_{n+1}^o A_n^o + b_{n+1}^e A_n^e + a_{n+1}^o B_n^{(1)o} + a_{n+1}^e B_n^{(1)e} \\ B_{n+1}^o = b_{n+1}^o B_n^e + b_{n+1}^e B_n^o + \gamma(T-Z)c_{n+1}^{(1)e}\ A_n^o\gamma(T+Z) + \\ \gamma(T-Z)c_{n+1}^{(1)o}\ A_n^e\gamma(T+Z) + \gamma(T+Z)a_{n+1}^{(1)o}\ C_n^e\gamma(T+Z) + \\ \gamma(T+Z)a_{n+1}^{(1)e}\ C_n^o\gamma(T+Z) \end{gathered} \tag{170}$$

$$
\begin{gathered}
B^e_{n+1} = b^o_{n+1} B^o_n + b^e_{n+1} B^e_n + \gamma(T-Z) c^{(1)o}_{n+1}\ A^o_n \gamma(T+Z) + \\
\gamma(T-Z) c^{(1)e}_{n+1}\ A^e_n \gamma(T+Z) + \gamma(T+Z) a^{(1)o}_{n+1}\ C^o_n \gamma(T+Z) + \\
\gamma(T+Z) a^{(1)e}_{n+1}\ C^e_n \gamma(T+Z) \\
C^o_{n+1} = c^o_{n+1} B^{(2)e}_n + c^e_{n+1} B^{(2)o}_n + b^o_{n+1} C^e_n + b^e_{n+1} C^o_n \\
C^e_{n+1} = c^o_{n+1} B^{(2)o}_n + c^e_{n+1} B^{(2)e}_n + b^o_{n+1} C^o_n + b^e_{n+1} C^e_n
\end{gathered}
$$

From these decompositions it is not difficult to check that the properties are also true for the rank $n+1$, QED.

5.6 Helicity transitions in connexion with mass terms

It is well known that in massless QED or QCD, transition amplitudes are helicity conserving. This is due to the vector nature of the gauge particles, photon or gluon. As is currently observed, helicity flips are due to mass terms. The same kind of rule should be expected for the presently studied impact factors. We would like to point out here that

- amplitudes that conserve lepton helicity (HC amplitudes) are even with respect to the lepton mass (i.e. they do not change their sign when $m \to -m$) ;
- amplitudes that do not conserve lepton helicity (HNC amplitudes) are odd with respect to the lepton mass (i.e. they change their sign when $m \to -m$).

Let us examine from this point of view the structure of (157). Each term in brackets may be again divided into a part (e) and a part (o). For example, consider the first term. The spinor β_1 is odd and does not involve a flip in helicity ($\theta_1 = 0$). We have

$$
\begin{gathered}
\bar\beta_3 \bar A_\ell \gamma(T-Z) A_r \beta_1 = \bar\beta^o_3 \bar A^e_\ell \gamma(T-Z) A^e_r \beta^o_1 + \bar\beta^o_3 \bar A^o_\ell \gamma(T-Z) A^o_r \beta^o_1 + \\
\bar\beta^e_3 \bar A^e_\ell \gamma(T-Z) A^o_r \beta^o_1 + \bar\beta^e_3 \bar A^o_\ell \gamma(T-Z) A^e_r \beta^o_1 + \\
\bar\beta^e_3 \bar A^e_\ell \gamma(T-Z) A^e_r \beta^o_1 + \bar\beta^e_3 \bar A^o_\ell \gamma(T-Z) A^o_r \beta^o_1 + \\
\bar\beta^o_3 \bar A^e_\ell \gamma(T-Z) A^o_r \beta^o_1 + \bar\beta^o_3 \bar A^o_\ell \gamma(T-Z) A^e_r \beta^o_1
\end{gathered} \tag{171}
$$

From the helicity properties of matrix factors A^e and A^o on one hand, and helicity properties of $\bar\beta^e_3$ and $\bar\beta^o_3$ on the other hand (see appendix 6.1), one can easily derive that the four first terms in the above expansion are helicity conserving and are even functions of the lepton mass ; the last four terms are odd functions of the lepton mass and induce a flip in helicity. An analogous analysis can be carried out in the same way for all terms in (157) ; whence the above stated properties.

6 Appendix

6.1 About the spinors U_3 of the outgoing lepton

From (6) and (7) the spinors U_3 of the outgoing lepton are given by

$$\begin{aligned} U_3^{\lambda_3} &= \sqrt{m}\ \mathcal{S}_3\ U_0^{\lambda_3} \\ \text{with} \quad \mathcal{S}_3 &= \mathcal{R}_Z(\phi_3)\mathcal{R}_Y(\theta_3)\mathcal{H}_Z(\chi_3) \end{aligned} \tag{A.1}$$

(with normalisation $\bar{U}_3 U_3 = 2m$).

First, we have

$$\mathcal{H}_Z(\chi_3)U_0^\lambda = \frac{1}{2}\exp(\frac{\chi_3}{2})\,[\gamma(T+Z) + \exp(-\chi_3)\gamma(T-Z)]\,U_0^\lambda \tag{A.2}$$

Secondly, taking into account that $S_Y\gamma(T\pm Z) = \gamma(T\mp Z)S_Y$, applying the rotation $\mathcal{R}_Y(\theta_3)$ to $\gamma(T\pm Z)$ yields

$$\mathcal{R}_Y(\theta_3)\gamma(T\pm Z) = \cos(\theta_3/2)\gamma(T\pm Z) - 2i\sin(\theta_3/2)\gamma(T\mp Z)S_Y \tag{A.3}$$

Then

$$\begin{aligned} \mathcal{R}_Z(\phi_3)\mathcal{R}_Y(\theta_3)\gamma(T\pm Z) = \cos(\theta_3/2)\gamma(T\pm Z)\mathcal{R}_Z(\phi_3) + \\ -2i\sin(\theta_3/2)\gamma(T\mp Z)S_Y\mathcal{R}_Z(-\phi_3) \end{aligned} \tag{A.4}$$

so that

$$\begin{aligned} U_3^\lambda = \sqrt{m}\ \mathcal{S}_3\ U_0^\lambda = \sqrt{m}\ \tfrac{1}{2}\exp(\frac{\chi_3}{2})\times \\ \{\cos(\theta_3/2)\exp(-i\lambda\phi_3)\,[\gamma(T+Z) + \exp(-\chi_3)\gamma(T-Z)] + \\ -2i\sin(\theta_3/2)\exp(i\lambda\phi_3)\,[\gamma(T-Z) + \exp(-\chi_3)\gamma(T+Z)]\,S_Y\}\ \ U_0^\lambda \end{aligned} \tag{A.5}$$

Eq. (A.5) can be nicely rewritten in the form

$$U_3^\lambda = \exp(-i\lambda\phi_3)\sqrt{\frac{E_3}{2}}\,[\gamma(T+Z)\alpha_3 + \gamma(T-Z)\beta_3] \tag{A.6}$$

with

$$\begin{aligned} \alpha_3 &= P(\frac{m}{E_3})\ \ \{\cos(\theta_3/2) - 2i\exp(2i\lambda\phi_3)\sin(\theta_3/2)\exp(-\chi_3)S_Y\}\ \ U_0^\lambda \\ &\text{where}\ \ P(\frac{m}{E_3}) = \frac{1}{2}\sqrt{\frac{2m}{E_3}}\ \exp(\chi_3/2), \ \ \text{and} \\ \beta_3 &= P(\frac{m}{E_3})\ \ \{\cos(\theta_3/2)\exp(-\chi_3) - 2i\exp(2i\lambda\phi_3)\sin(\theta_3/2)S_Y\}\ \ U_0^\lambda \end{aligned} \tag{A.7}$$

Remind that $\cosh(\chi_3/2) = \sqrt{(E_3/m+1)/2}$, so that

$$P(\frac{m}{E_3}) = \frac{1}{2}\left[\sqrt{1+m/E_3} + \sqrt{1-m/E_3}\right] \tag{A.8}$$

It is important to notice that $P(u)$ is an even function of $u = m/E_3$. Therefore, its development in powers of u contains even powers only and when $u \ll 1$ we may write

$$P(u) \approx 1 + O(u^2) \tag{A.9}$$

In the framework of jet-like kinematics approximation, this means that since it is sufficient to keep only terms of first order in the development of spinors, $P(\frac{m}{E_3})$ can be safely taken equal to 1. Then, since we have $E_3 \gg m$ and $\theta_3 \ll 1$, keeping only terms up to first order leads to the following approximations

$$\begin{aligned} \alpha_3 \approx\ U_0^\lambda, \quad \beta_3 \approx \left[\frac{m}{2E_3} - i\xi_3 S_Y\right]\ U_0^\lambda \\ \text{where} \quad \xi_3 = \exp(2i\lambda\phi_3)\theta_3 \end{aligned} \tag{A.10}$$

Here again, we note that the spinor β_3 to which the matrix $\gamma(T-Z)$ applies is one order less than the spinor α_3 to which the matrix $\gamma(T+Z)$ applies : β_3 is of order 1 while α_3 is of order 0.

It is also worth noticing the following general properties of spinors one can derive from (A.7). Each of the two spinors α_3 and β_3 is made up of two pieces that have different helicity properties and different behaviors with respect to the lepton mass. Thus, taking into account the relation

$$P(\frac{m}{E_3})\exp(-\chi_3) = \frac{1}{2}\left[\sqrt{1+m/E_3} - \sqrt{1-m/E_3}\right] \tag{A.11}$$

we observe that

- in α_3, the piece that has the same helicity as that of U_0^λ is an even function of the lepton mass m, while the part $\propto -iS_Y U_0^\lambda = \lambda U_0^{-\lambda}$ that has an opposite helicity is an odd function of m ;
- on the contrary, in β_3, the part with no helicity-flip is an odd function of m, while the part with helicity-flip is an even function of m.

6.2 On "mass-parity" of QED amplitudes

Any QED amplitude has one of the following two forms ; either

$$\bar{U}_\ell^{\lambda_\ell}\, \mathcal{T}\, U_k^{\lambda_k} \tag{B.1}$$

for a subprocess like $\mathrm{lepton}_k \to X + \mathrm{lepton}_\ell$ where both leptons are of the same species, or

$$\bar{U}_\ell^{\lambda_\ell}\, \mathcal{T}\, W_k^{\lambda_k} \tag{B.2}$$

for a subprocess where a lepton pair is produced, where in that case U is the spinor of the lepton and W the spinor associated with its antiparticle. In both cases, $\mathcal{T}$ is a 4X4 transition matrix which is a succession of products of lepton propagator and γ-matrices describing vertices.

In order to eliminate the mass terms from the various numerators of propagators entering into $\mathcal{T}$, we may apply Dirac equation as many times as necessary. In this way, the effective matrix $\mathcal{T}$ is found as a linear combination of products of γ-matrices, each of these products containing an odd number of γ-matrices, due to the vector nature of the lepton-photon coupling. Therefore, $\mathcal{T}$ may be expressed in the quite general form

$$\mathcal{T} \sim \gamma(A) + \gamma(B)\gamma_5 \tag{B.3}$$

where A and B are 4-vectors depending on 4-momenta of particles taking part in the subprocess, and on polarization 4-vectors of possible outgoing photons. Since it is always possible to make the choice[5]

$$W^\lambda \equiv V^\lambda = \gamma_5\, U^\lambda \tag{B.4}$$

one then sees that to study the "mass-parity" property of more general amplitudes, it is sufficient to consider the elementary matrix elements

$$\bar{U}_\ell^{\lambda_\ell}\, \gamma_\mu\, U_k^{\lambda_k}\ , \quad \bar{U}_\ell^{\lambda_\ell}\, \gamma_\mu \gamma_5\, U_k^{\lambda_k} \tag{B.5}$$

In fact, it appears more convenient to consider instead matrix elements

$$\bar{U}_\ell^{\lambda_\ell}\, \gamma_\mu\, (1 \pm \gamma_5)\; U_k^{\lambda_k} \tag{B.6}$$

Using the definition (8), we get

$$(1 \pm \gamma_5)\; U^\lambda = \exp\left(\pm \lambda \chi\right)\, (1 \pm \gamma_5)\; U'^\lambda \quad \text{with}$$

[5]Which is different from that in Eq. (70).

$$U'^{\lambda} = \sqrt{m}\left[\exp(-i\lambda\varphi)\cos(\frac{\theta}{2})U_0^{\lambda} + 2\lambda\exp(i\lambda\varphi)\sin(\frac{\theta}{2})U_0^{-\lambda}\right] \tag{B.7}$$

Then,

$$\bar{U}_\ell^{\lambda_\ell}\,\gamma_\mu\,(1\pm\gamma_5)\,U_k^{\lambda_k} = m\,\exp(\pm[\lambda_\ell\chi_\ell+\lambda_k\chi_k])\,\mathcal{V}_\mu(\theta_l,\varphi_\ell;\theta_k,\varphi_k) \tag{B.8}$$

where

$$\mathcal{V}_\mu(\theta_l,\varphi_\ell;\theta_k,\varphi_k) = \left[\exp(i\lambda_\ell\varphi_\ell)\cos(\frac{\theta_\ell}{2})\bar{U}_0^{\lambda_\ell} + 2\lambda_\ell\exp(-i\lambda_\ell\varphi_\ell)\sin(\frac{\theta}{2})\bar{U}_0^{-\lambda_\ell}\right]\times$$
$$\gamma_\mu\,(1\pm\gamma_5)\left[\exp(-i\lambda_k\varphi_k)\cos(\frac{\theta_k}{2})U_0^{\lambda_k} + 2\,\lambda_k\exp(i\lambda_k\varphi_k)\sin(\frac{\theta_k}{2})U_0^{-\lambda_k}\right] \tag{B.9}$$

clearly does not depend on the mass m. Hence, the whole dependence of amplitudes (B.6) on m is entirely contained in the factors

$$\mathcal{P}^{(\pm)}(\ell,\lambda_\ell\,;k,\lambda_k) = m\,\exp(\pm[\lambda_\ell\chi_\ell+\lambda_k\chi_k]) \tag{B.10}$$

Using

$$\exp(\pm\lambda\chi) = \frac{1}{\sqrt{2}}\sqrt{\frac{E}{m}}\left[\sqrt{1+\frac{m}{E}}\pm 2\lambda\sqrt{1-\frac{m}{E}}\;\right]$$

we find

$$\mathcal{P}^{(\pm)}(\ell,\lambda_\ell\,;k,\lambda_k) = \frac{1}{2}\sqrt{E_\ell E_k}\,\left[\sqrt{1+\frac{m}{E_\ell}}\pm 2\lambda_\ell\sqrt{1-\frac{m}{E_\ell}}\;\right]\times$$
$$\left[\sqrt{1+\frac{m}{E_k}}\pm 2\lambda_k\sqrt{1-\frac{m}{E_k}}\;\right] \tag{B.11}$$

It is then clear that only expressions (B.11) for which $\lambda_\ell = \lambda_k$ are even function of m, whereas those for which $\lambda_\ell = -\lambda_k$ are odd functions of m. It is worth noticing here that 4-momenta of particles entering the composition of A or B in (B.3) are even functions of m (through energies $E = \sqrt{m^2+\vec{p}^2}$). Therefore, we may draw the following conclusion.

Let us consider one of the Feynman diagrams describing a given QED process where, to simplify, only one species of leptons is assumed to be involved. In the corresponding amplitude, any lepton ℓ_1 (any anti-lepton $\bar{\ell}_1$) is connected, either to another lepton ℓ_2 (another anti-lepton $\bar{\ell}_2$) through a sequence of lepton propagators, or to an anti-lepton $\bar{\ell}_3$ (lepton ℓ_3) through, for example, subprocesses $\gamma^\star \leftrightarrow \ell+\bar{\ell}$ or $\gamma_1+\gamma_2 \leftrightarrow \ell+\bar{\ell}$ with real or virtual photons. We may thus associate all leptons and anti-leptons taking part in the process in binomials like (ℓ_1, ℓ_2), $(\bar{\ell}_1, \bar{\ell}_2)$ and $(\ell, \bar{\ell})$. The "mass-parity" of the amplitude of the diagram is then found as follows. Let N_e the number of binomials having particles with the same helicities, N_o that of binomials where particles have opposite helicities. Then, the "mass-parity" of the amplitude is equal to that of N_o : even is N_o is even, odd if N_o is odd. We know that the all set of Feynman diagrams describing the process under consideration can be simply obtained from that particular diagram by interchanging lines of leptons or lines of anti-leptons, or interchanging between them in an appropriate way lepton lines with anti-lepton lines. But it is clear that such operations lead to new amplitudes that possess the same "mass-parity". We thus conclude that the full amplitude describing a process for given helicities of particles has the same "mass-parity" as that given by any of the underlying Feynman diagrams. This does not mean however that all sub-amplitudes are of the same order with regard to the mass m. For example, we may find $N_o = 0$ for some diagram, $N_o = 2$ for another one. This implies that the amplitude of the latter is $\propto m^2$ compared to that of the first one.

6.3 Some useful formulas

In Ref [**4**] useful relations involving Dirac spinors are given. It can be shown that for any spinor we have

$$\gamma_\mu \; U^\lambda = t_\mu \; U^\lambda - 2\lambda \; z_\mu \; V^\lambda + 2 \; \lambda \; \sqrt{2} \; \epsilon_\mu^{(2\lambda)} \; V^{-\lambda} \tag{C.1}$$

where t, z and circular polarisations $\epsilon^{(2\lambda)}$ with $\lambda = \pm 1/2$ are defined by eqs (2) and (3). Hence,

$$\gamma(\epsilon^{(\Lambda)\star}) \; U^\lambda = -2 \; \lambda \; \sqrt{2} \; \delta_{\Lambda, 2\lambda} \; V^{-\lambda} \tag{C.2}$$

In particular, for the ingoing lepton we may set $\epsilon^{(\Lambda)} = E^{(\Lambda)}$ (see (2)) and

$$\gamma(E^{(\Lambda)\star}) \; U_1^\lambda = -2 \; \lambda \; \sqrt{2} \; \delta_{\Lambda, 2\lambda} \; V_1^{-\lambda} \tag{C.3}$$

Defining, for any 4-vector Q

$$Q_T^{(\Lambda)} = Q_X + i\Lambda Q_Y = \Lambda\sqrt{2} E^{(\Lambda)} \cdot Q \tag{C.4}$$

we also get

$$\gamma(Q_T) \; U_0^\lambda = Q_T^{(2\lambda)} \; V_0^{-\lambda} \tag{C.5}$$

$$\gamma(Q_T) \; \gamma(E^{(\Lambda)\star}) \; U_0^\lambda = 2 \; \lambda \; \sqrt{2} \; \delta_{\Lambda, 2\lambda} \; Q_T^{(-2\lambda)} \; U_0^\lambda \tag{C.6}$$

We have

$$\gamma(T - Z) \; U_0^\lambda = U_0^\lambda - 2 \; \lambda \; V_0^\lambda \tag{C.7}$$

and

$$\gamma(E^{(\Lambda)\star})\ \gamma(T-Z)\ U_0^\lambda = -\sqrt{2}\ \delta_{\Lambda,2\lambda}\ \left[\ U_0^{-\lambda} + 2\ \lambda\ V_0^{-\lambda}\ \right] \tag{C.8}$$

$$\gamma(E^{(\Lambda)\star})\ \gamma(T-Z)\ V_0^\lambda = -\sqrt{2}\ \delta_{\Lambda,2\lambda}\ \left[\ V_0^{-\lambda} + 2\ \lambda\ U_0^{-\lambda}\ \right] \tag{C.9}$$

In (147) we have the matrix $a^{(1)}\gamma(T-Z) = Q_+\gamma(E^{(\Lambda)\star})\ \gamma(T-Z)/2$. Its action on the basis of spinors $U_0^\lambda,\ V_0^\lambda$ may be summed up by the following matrices :

$$a^{(1)}\gamma(T-Z) \equiv \frac{Q_+}{2}\ \sqrt{2}\ \begin{pmatrix} 0 & 0 & 0 & 0 \\ -1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & -1 & 0 \end{pmatrix} \quad \text{for} \quad \Lambda = +1 \tag{C.10}$$

$$a^{(1)}\gamma(T-Z) \equiv \frac{Q_+}{2}\ \sqrt{2}\ \begin{pmatrix} 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{for} \quad \Lambda = -1 \tag{C.11}$$

where lines as well as columns are arranged according to the sequence $U_0^\uparrow,\ U_0^\downarrow,\ V_0^\uparrow,\ V_0^\downarrow$. We also have

$$\gamma(Q_T)\ \gamma(E^{(\Lambda)\star}) \equiv Q_T^{(-)}\ \sqrt{2}\ \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{for} \quad \Lambda = +1 \tag{C.12}$$

$$\gamma(Q_T)\ \gamma(E^{(\Lambda)\star}) \equiv Q_T^{(+)}\ \sqrt{2}\ \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \quad \text{for} \quad \Lambda = -1 \tag{C.13}$$

Thus, the matrix $b^{(1)}$ in (147) may be represented in the form :

$$b^{(1)} \equiv \sqrt{2}\ \begin{pmatrix} Q_T^{(-)} & 0 & 0 & 0 \\ 0 & 0 & -m & 0 \\ 0 & 0 & Q_T^{(-)} & 0 \\ m & 0 & 0 & 0 \end{pmatrix} \quad \text{for} \quad \Lambda = +1 \tag{C.14}$$

$$b^{(1)} \equiv -\ \sqrt{2}\ \begin{pmatrix} 0 & 0 & 0 & -m \\ 0 & Q_T^{(+)} & 0 & 0 \\ 0 & m & 0 & 0 \\ 0 & 0 & 0 & Q_T^{(+)} \end{pmatrix} \quad \text{for} \quad \Lambda = -1 \tag{C.15}$$

We have

$$\gamma(T-Z)\ \gamma(T+Z) = 2\,(1-\gamma(Z)\gamma(T)) = 2\,(1-2\gamma_5 S_Z) \tag{C.16}$$

so that

$$\gamma(T-Z)\ \gamma(T+Z)\ U_0^\lambda = 2\left(U_0^\lambda - 2\lambda V_0^\lambda\right) \tag{C.17}$$

Thus

$$\gamma(T-Z)\ \gamma(T+Z) \equiv\ 2\ \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 1 \\ -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix} \tag{C.18}$$

Let us define

$$\eta^{(\mp)} = \tan(\frac{\theta}{2})\exp(\mp i\phi) \quad \text{and} \quad \zeta^{(\mp)} = \frac{Q_+}{2}\ \eta^{(\mp)} \tag{C.19}$$

then, the matrix $b^{(2)}$ in (147) has the form :

$$b^{(2)} \equiv \sqrt{2}\ \begin{pmatrix} Q_T^{(-)} - \zeta^{(-)} & 0 & \zeta^{(-)} & 0 \\ 0 & -\zeta^{(-)} & -m & -\zeta^{(-)} \\ \zeta^{(-)} & 0 & Q_T^{(-)} - \zeta^{(-)} & 0 \\ m & -\zeta^{(-)} & 0 & -\zeta^{(-)} \end{pmatrix} \quad \text{for} \quad \Lambda = +1 \tag{C.20}$$

$$b^{(2)} \equiv -\ \sqrt{2}\ \begin{pmatrix} -\zeta^{(+)} & 0 & \zeta^{(+)} & -m \\ 0 & Q_T^{(+)} - \zeta^{(+)} & 0 & -\zeta^{(+)} \\ \zeta^{(+)} & m & -\zeta^{(+)} & 0 \\ 0 & -\zeta^{(+)} & 0 & Q_T^{(+)} - \zeta^{(+)} \end{pmatrix} \quad \text{for} \quad \Lambda = -1 \tag{C.21}$$

We have

$$\gamma(T+Z)\ U_0^\lambda = U_0^\lambda + 2\ \lambda\ V_0^\lambda \tag{C.22}$$

$$\gamma(T+Z)\ V_0^\lambda = -V_0^\lambda - 2\ \lambda\ U_0^\lambda \tag{C.23}$$

and

$$\gamma(E^{(\Lambda)\star})\ \gamma(T+Z)\ U_0^\lambda = \sqrt{2}\ \delta_{\Lambda,2\lambda}\ \left[\ U_0^{-\lambda} - 2\ \lambda\ V_0^{-\lambda}\ \right] \tag{C.24}$$

$$\gamma(E^{(\Lambda)\star})\ \gamma(T+Z)\ V_0^\lambda = \sqrt{2}\ \delta_{\Lambda,2\lambda}\ \left[\ V_0^{-\lambda} - 2\ \lambda\ U_0^{-\lambda}\ \right] \tag{C.25}$$

$$\gamma(Q_T)\ \gamma(T+Z)\ U_0^\lambda = Q_T^{(2\lambda)}\ \left[\ V_0^{-\lambda} - 2\ \lambda\ U_0^{-\lambda}\ \right] \tag{C.26}$$

$$\gamma(Q_T)\ \gamma(T+Z)\ V_0^\lambda = Q_T^{(2\lambda)}\ \left[\ U_0^{-\lambda} - 2\ \lambda\ V_0^{-\lambda}\ \right] \tag{C.27}$$

then, the matrix $c^{(1)}\gamma(T+Z)$ in (147) has the form :

$$c^{(1)}\ \gamma(T+Z) \equiv \frac{\sqrt{2}}{2}\ \eta^{(-)} \begin{pmatrix} -m & Q_T^{(-)} & m & Q_T^{(-)} \\ -Q_T^{(+)} & -m & Q_T^{(+)} & -m \\ -m & Q_T^{(-)} & m & Q_T^{(-)} \\ Q_T^{(+)} & m & -Q_T^{(+)} & m \end{pmatrix} +$$

$$\frac{\sqrt{2}}{2}\ Q_- \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \end{pmatrix} \quad \text{for} \quad \Lambda = +1 \qquad \text{(C.28)}$$

and

$$c^{(1)}\ \gamma(T+Z) \equiv \frac{\sqrt{2}}{2}\ \eta^{(+)} \begin{pmatrix} m & -Q_T^{(-)} & -m & -Q_T^{(-)} \\ Q_T^{(+)} & m & -Q_T^{(+)} & m \\ m & -Q_T^{(-)} & -m & -Q_T^{(-)} \\ -Q_T^{(+)} & -m & Q_T^{(+)} & -m \end{pmatrix} +$$

$$\frac{\sqrt{2}}{2}\ Q_- \begin{pmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{for} \quad \Lambda = -1 \qquad \text{(C.29)}$$

From these relations we derive the following expressions of the 8X8 matrices $\mathcal{Y}_r$ and $\mathcal{Y}_l$

$$\mathcal{Y}_r \equiv \frac{\sqrt{2}}{2} \times$$

$$\begin{pmatrix} 2Q_T^{(-)} - 2\zeta^{(-)} & 0 & 2\zeta^{(-)} & 0 & 0 & 0 & 0 & 0 \\ 0 & -2\zeta^{(-)} & -2m & -2\zeta^{(-)} & -Q_+ & 0 & -Q_+ & 0 \\ 2\zeta^{(-)} & 0 & 2Q_T^{(-)} - 2\zeta^{(-)} & 0 & 0 & 0 & 0 & 0 \\ 2m & -2\zeta^{(-)} & 0 & -2\zeta^{(-)} & -Q_+ & 0 & -Q_+ & 0 \\ -m\eta^{(-)} & \eta^{(-)}Q_T^{(-)} & m\eta^{(-)} & \eta^{(-)}Q_T^{(-)} & 2Q_T^{(-)} & 0 & 0 & 0 \\ Q_- - \eta^{(-)}Q_T^{(+)} & -m\eta^{(-)} & \eta^{(-)}Q_T^{(+)} - Q_- & -m\eta^{(-)} & 0 & 0 & -2m & 0 \\ -m\eta^{(-)} & \eta^{(-)}Q_T^{(-)} & m\eta^{(-)} & \eta^{(-)}Q_T^{(-)} & 0 & 0 & 2Q_T^{(-)} & 0 \\ \eta^{(-)}Q_T^{(+)} - Q_- & m\eta^{(-)} & Q_- - \eta^{(-)}Q_T^{(+)} & m\eta^{(-)} & 2m & 0 & 0 & 0 \end{pmatrix}$$

$$\text{for} \quad \Lambda = +1 \qquad \text{(C.30)}$$

$$\mathcal{Y}_r \equiv \frac{\sqrt{2}}{2} \times$$

$$\begin{pmatrix} 2\zeta^{(+)} & 0 & -2\zeta^{(+)} & 2m & 0 & -Q_+ & 0 & Q_+ \\ 0 & 2(\zeta^{(+)} - Q_T^{(+)}) & 0 & 2\zeta^{(+)} & 0 & 0 & 0 & 0 \\ -2\zeta^{(+)} & -2m & 2\zeta^{(+)} & 0 & 0 & Q_+ & 0 & -Q_+ \\ 0 & 2\zeta^{(+)} & 0 & 2(\zeta^{(+)} - Q_T^{(+)}) & 0 & 0 & 0 & 0 \\ m\eta^{(+)} & Q_- - \eta^{(+)}Q_T^{(-)} & -m\eta^{(+)} & Q_- - \eta^{(+)}Q_T^{(-)} & 0 & 0 & 0 & 2m \\ \eta^{(+)}Q_T^{(+)} & m\eta^{(+)} & -\eta^{(+)}Q_T^{(+)} & m\eta^{(+)} & 0 & -2Q_T^{(+)} & 0 & 0 \\ m\eta^{(+)} & Q_- - \eta^{(+)}Q_T^{(-)} & -m\eta^{(+)} & Q_- - \eta^{(+)}Q_T^{(-)} & 0 & -2m & 0 & 0 \\ -\eta^{(+)}Q_T^{(+)} & -m\eta^{(+)} & \eta^{(+)}Q_T^{(+)} & -m\eta^{(+)} & 0 & 0 & 0 & -2Q_T^{(+)} \end{pmatrix}$$

$$\text{for} \quad \Lambda = -1 \tag{C.31}$$

$$\mathcal{Y}_l^{\Lambda} = -\Gamma_0 \left[\mathcal{Y}_r^{-\Lambda}\right]^{\dagger} \Gamma_0 \quad \text{with} \quad \Gamma_0 = \begin{pmatrix} \gamma_0 & 0 \\ 0 & \gamma_0 \end{pmatrix} \tag{C.32}$$

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