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ON THE SPINOR STRUCTURE OF THE PROTON WAVE FUNCTION

Christian CARIMALO

Laboratoire de Physique Corpusculaire

Collège de France, IN2P3-CNRS

11, Place Marcelin Berthelot, F-75231 Paris Cedex 05, France

ABSTRACT

In the framework of a relativistic extension of the three constituent-quark model of the proton, spinors of third rank in Dirac indices are used to express covariantly the spin structure of S-state baryons of spin 1/2 or 3/2. These spinors are defined directly in terms of 4-spinors of the Dirac space D, and classified according to the irreducible representations of both the (internal) Lorentz spin group acting on $E = D \times D \times D$, and the permutation group S_3. A particular emphasis is placed on *genuine* spin-1/2 wave functions belonging to representations $D(0, \frac{1}{2}) \oplus D(\frac{1}{2}, 0)$ in E. They might be the fundamental spin-1/2 wave functions from which that of the proton should be constructed, and are possibly related to the *vertex function* for the transition $3\, q \to p$, rather than to the associated Bethe-Salpeter amplitude. All possible combinations of spin, flavor and momentum distributions of quarks in S-state baryons are considered from the point of view of S_3. We also suggest the possibility of defining Dirac matrices acting on E, which may prove important in view of studying three-quark confinement forces through an effective relativistic wave equation for the proton.

Keywords : Quark model, Baryons, Particle Structure, Wave Equations.

1. INTRODUCTION

The relativistic extension of the fruitful ideas of the non-relativistic quark-model has always run into the problem of how the wave function of a composite hadron (defined as a multiquark state) should be Lorentz-transformed from the hadron rest frame. In fact, the true problem is to find the appropriate relativistic description of spin states of bound quarks and the way these states combine to yield the correct covariance of the hadron wave function. An immediate and minimal recipe consists in simply replacing the Pauli spinors involved in non-relativistic $SU(6)$ hadron wave functions by Dirac spinors, with the advantage of recovering the non-relativistic model in the static limit. For S state (low-lying) hadrons, one is thus led to covariant wave functions of the Bargmann-Wigner type [**1**] which have been recently revisited in Ref. [**2**]. There, these wave functions have been shown to be the basic spinor forms that are used, as a zeroth-order approximation, in practically all covariant approaches of hadronic wave functions, whenever a weak-binding or collinearity approximation is assumed in the momentum distribution of bound quarks. In this respect, numerous models then appear equivalent, at least in their treatment of spinor wave functions, their only differences lying in the explicit momentum distributions of quarks they respectively use.

According to that formalism, the three-quark nucleon wave function is described by means of the following third-rank spinor [**2**, **3**, **4**] :

$$\varphi_{[\alpha\beta]\gamma}(P) = ((M+ \not{P})\gamma_5\mathcal{C})_{\alpha\beta} U_\gamma(P) \tag{1.1}$$

where P and M are the four-momentum and the mass of the nucleon respectively; $\alpha,\ \beta,\ \gamma$ are four-valued Dirac indices, $\mathcal{C}$ is the usual 4×4 conjugation matrix and $U(P)$ is a Dirac spinor satisfying the Dirac equation $(M- \not{P})U(P) = 0$ [**5**].

That the spinor form (1.1) is a simple relativistic extension of a usual non-relativistic one becomes more clear when expressing it in terms of Dirac spinors, since one has the identity :

$$\varphi_{[\alpha\beta]\gamma}(P) = \left(U^{\uparrow}{}_\alpha(P)U^{\downarrow}{}_\beta(P) - U^{\downarrow}{}_\alpha(P)U^{\uparrow}{}_\beta(P)\right) U_\gamma(P) \tag{1.2}$$

which allows one an easy check of the Bargmann-Wigner equations :

$$(M- \not{P})_\alpha{}^{\alpha'}\varphi_{[\alpha'\beta]\gamma}(P) = (M- \not{P})_\beta{}^{\beta'}\varphi_{[\alpha\beta']\gamma}(P) = (M- \not{P})_\gamma{}^{\gamma'}\varphi_{[\alpha\beta]\gamma'}(P) = 0 \tag{1.3}$$

as well as the symmetry relations :

$$\varphi_{[\alpha\beta]\gamma}(P) = -\varphi_{[\beta\alpha]\gamma}(P) \tag{1.4}$$

and

$$\varphi^{(\uparrow,\downarrow)}_{[\alpha\beta]\gamma}(P) + \varphi^{(\uparrow,\downarrow)}_{[\gamma\alpha]\beta}(P) + \varphi^{(\uparrow,\downarrow)}_{[\beta\gamma]\alpha}(P) = 0 \tag{1.5}$$

In (1.5), the label $(\uparrow,\downarrow)$ refers to the helicity assigned to the spinor $U_\kappa(P)$ in $\varphi_{[\delta\zeta]\kappa}(P)$.

Let us notice that Eq.(1.3-5) are usually considered, in the constituent-quark model, as the (necessary) conditions for a third-rank spinor $\varphi_{\alpha\beta\gamma}$ to represent a spin 1/2 composite particle [**3**].

The spinor form (1.1-2) has been used in particular by the authors of Refs. [**6**] and [**7**] to compute the nucleon electromagnetic form factors, in the framework of the relativistic harmonic oscillator model. As shown by R. G. Lipes in [7], this model leads to a satisfactory fit of the magnetic proton form factor G_M, provided the expression obtained for G_M is divided, in a somewhat arbitrary way, by the quantity $\chi = 1+Q^2/4M^2$ ($-Q^2$ being the momentum transfer squared), especially in order to recover the experimentally observed Q^{-4} power fall-off of G_M. He also obtained better agreement with electroproduction data after dividing by χ.

This rather uncomfortable feature might have suggested, at first sight, a failure of the four-dimensional harmonic oscillator model itself. Things are not so simple however, as pointed out for the first time by the authors of Ref. [**8**] who showed that the proton spinor wave function can be described through third-rank spinors different from (1.1) and such that, within the same quark-interaction model, the extra factor χ does not appear at all in the calculated expression of G_M. How an extra power of Q^2 may thus appear or not in G_M, depending on the choice made for the proton spinor wave function, we explain it as follows.

In constructing the three-quark proton wave function with Dirac spinors instead of Pauli spinors, one is led in fact to search for spin 1/2 wave functions in the 64-dimensional space $E = D \times D \times D$ where D is the 4-dimensional space of Dirac spinors. Vectors of E are Lorentz-transformed by operators which are generated by that part of the total angular momentum of the proton representing the spin angular momentum of that particle. From the point of view of the corresponding induced Lorentz group, which may be called the Lorentz spin group, each component space D in the direct product space E is to be considered as a representation $D(0,\frac{1}{2}) \oplus D(\frac{1}{2},0)$ of the Lorentz group, while E itself is to be regarded as a representation space of that group, which is reducible into a direct sum of irreducible representation spaces $D(j_1, j_2)$ of the proper Lorentz group, according to the formula :

$$E = 5\left\{D(0,\frac{1}{2}) \oplus D(\frac{1}{2},0)\right\} \oplus 3\left\{D(\frac{1}{2},1) \oplus D(1,\frac{1}{2})\right\} \oplus D(\frac{3}{2},0) \oplus D(0,\frac{3}{2}) \tag{1.6}$$

where it is shown that the irreducible representations of the full Lorentz group $D(0,\frac{1}{2}) \oplus D(\frac{1}{2},0)$, $D(\frac{1}{2},1) \oplus D(1,\frac{1}{2})$ and $D(\frac{3}{2},0) \oplus D(0,\frac{3}{2})$ appear in this sum 5 times, 3 times and once respectively [**9**].

It is well known that, in its turn, each space $D(j_1, j_2)$ reduces with respect to the rotation group into the direct sum :

$$D(j_1, j_2) = \bigoplus_{J=|j_1-j_2|}^{J=j_1+j_2} D(J) \tag{1.7}$$

where $D(J)$ is an irreducible representation of the rotation group for spin J. Thus, in E, spin 1/2 wave functions are to be found in representations $D(0, \frac{1}{2}) \oplus D(\frac{1}{2}, 0)$ and $D(1, \frac{1}{2}) \oplus D(\frac{1}{2}, 1)$. Remember that under a Lorentz transformation $\mathcal{L}$, a wave function $\Psi^{j_1,j_2}_{m_1,m_2}$ (with $m_1 = -j_1,\ -j_1+1,\ ...,\ j_1$ and $m_2 = -j_2,\ -j_2+1,\ ...,\ j_2$) that belongs to a representation $D(j_1, j_2)$ transforms as :

$$\Psi'^{j_1,j_2}_{m'_1,m'_2} = \sum_{m_1=-j_1}^{m_1=j_1} \sum_{m_2=-j_2}^{m_2=j_2} \mathcal{D}^{j_1}_{m_1,m'_1}(L) \mathcal{D}^{j_2}_{m_2,m'_2}(L^{\dagger -1}) \Psi^{j_1,j_2}_{m_1,m_2} \tag{1.8}$$

where L is the $SL(2C)$-representation of $\mathcal{L}$ and the $\mathcal{D}^j_{m,m'}$ are standard Wigner matrices.

Now, to compute form factors arising from the elementary process $\gamma^\star + p \to p$, one has to connect the spinor wave function of the outgoing proton to that of the ingoing proton by a Lorentz boost $\mathcal{L}$ acting on E. The rapidity y of that transformation is defined through the relation :

$$\cosh y \equiv \frac{P'.P}{M^2} = 1 + \frac{Q^2}{2M^2} = 2\chi - 1 \tag{1.9}$$

It follows from Eq.(1.8) that if the proton spinor wave function belongs to a subspace $D(0, \frac{1}{2}) \oplus D(\frac{1}{2}, 0)$ of E, it will transform just like an ordinary Dirac spinor, a feature that is implicitly assumed in assigning a Dirac 4-spinor to each of the asymptotically “free” ingoing and outgoing protons when writing the general form of the proton electromagnetic current. If, instead, the proton spinor wave function belongs to a representation $D(1, \frac{1}{2}) \oplus D(\frac{1}{2}, 1)$ in E, its transformation will then involve, as compared to the preceding case, additional matrix elements $\mathcal{D}^1_{m,m'}(L) = \delta_{m,m'} e^{my}$ and $\mathcal{D}^1_{m,m'}(L^{\dagger -1}) = \delta_{m,m'} e^{-my}$ (with $m = 0, \pm 1$) which, taking into account Eq.(1.9), finally cause the appearance of an extra power of Q^2 :

$$\Psi'^{(1,1/2)}_{\pm 1,\mp 1/2} = e^{\pm 3y/2} \Psi^{(1,1/2)}_{\pm 1,\mp 1/2} \quad , \quad \Psi'^{(1/2,1)}_{\mp 1/2,\pm 1} = e^{\pm 3y/2} \Psi^{(1/2,1)}_{\mp 1/2,\pm 1} \tag{1.10}$$

As will be shown in the following, the spinor form (1.1-2) does have a component in such a representation $D(1, \frac{1}{2}) \oplus D(\frac{1}{2}, 1)$. This is why its use in computing G_M

gives rise to the additional factor $\chi = \cosh^2(y/2)$[1]. This clearly demonstrates the very importance, already stressed in Ref. [**8**], of the choice of spinor wave functions in computing dynamical quantities in any relativistic-quark model. Let us also notice the striking fact that if the proton spinor wave function is chosen, as would seem natural, in a representation $D(0,\frac{1}{2}) \oplus D(\frac{1}{2},0)$ its expression will be very different from the simple one constructed from (1.1), with no known non-relativistic analogues. In particular, as will be seen below, it will necessarily involve negative-energy Dirac spinors (which may be defined as $V(P) = \gamma_5 U(P)$), so that the Bargmann-Wigner constraints (1.3) do not apply any more to such a wave function.

The necessity of introducing "small-components spinors" in order to get a better agreement with the data was already recognized in [10]. This is equivalent to introducing negative-energy spinors in the proton wave function and may be interpreted as revealing the fact that bound quarks should be treated as virtual rather than real particles, in the same way, for instance, as a virtual electron is described by a propagator involving both positive-energy and negative-energy Dirac spinors. On the other hand, keeping the form (1.1) to describe the proton wave function should imply severe constraints on the underlying dynamical model. In that respect, the model proposed by Brodsky and Lepage and by Chernyak and Zhitnitsky [**11**], in order to analyse exclusive processes at high momentum transfer, seems to work : in that model, though the spinor form (1.1) is used, the minimum number of exchanged gluons and quarks in each graph describing the electromagnetic process that leads to G_M is sufficiently large to restore the correct Q^{-4} fall-off of G_M (through the associated propagators). In the last section we will discuss the related question of the interpretation of the wave functions.

Let us add that choosing the wave function of a spin-3/2 low-lying baryon either in $D(1,\frac{1}{2}) \oplus D(\frac{1}{2},1)$ or in $D(\frac{3}{2},0) \oplus D(0,\frac{3}{2})$ would make no difference, as regards the powers of Q^2 arising from the Lorentz boost in (1.8) : these powers are the same for both representations. This renders the spin-1/2 case even more special.

In Ref. [**8**] tensor-spinors of third rank in Dirac indices have been studied in a matrix formalism and listed according to their Lorentz-covariance and their symmetry under the permutation group S_3 acting on the triplet $\{\alpha,\ \beta,\ \gamma\}$. By combining these generalized Rarita-Schwinger wave functions (as they were called by the authors) with quark momenta, one can, in principle, obtain the wave function of any three-quark state of definite spin and parity. In the present work our approach is somewhat different. Here, we intend to perform a systematic study of the structure of the space E with respect to the Lorentz spin group, as given by formula (1.6). In section 2 we start with defining the Casimir operators that are associ-

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[1] As a comparison, if $\mathcal{L}$ were a rotation of angle θ around the z-axis, then $\mathcal{D}^j_{m,m'}(L) = \mathcal{D}^j_{m,m'}(L^{\dagger-1}) = \delta_{m,m'} e^{im\theta}$, $\Psi'^{(1,1/2)}_{\pm1,\mp1/2} = e^{\pm i\theta/2}\Psi^{(1,1/2)}_{\pm1,\mp1/2}$, $\Psi'^{(1/2,1)}_{\mp1/2,\pm1} = e^{\pm i\theta/2}\Psi^{(1/2,1)}_{\mp1/2,\pm1}$ and in that case there would be no difference between the two above-mentioned representations.

ated with the spin angular momentum of the proton, and the diagonalization of which leads to the reduction rule (1.6). In section 3 we derive a 64-dimensional basis of eigenspinors of those operators, which we classify according to both (1.6) and irreducible representations of S_3. In section 4 we analyse, with respect to S_3, the general structure of the Lorentz-invariant momentum distribution of quarks in the proton and all its possible combinations with spin and flavor distributions. In section 5 we suggest possible constructions of Dirac matrices acting on E, and point out their usefulness in the framework of an effective relativistic wave equation for the proton. Section 6 contains some concluding remarks, regarding the interpretation of the spinor wave functions here considered.

2. THE SPIN ANGULAR MOMENTUM OF THE PROTON IN THE CONSTITUENT-QUARK MODEL

We assume here that the effective degrees of freedom making up the proton state are those of three constituent quarks. Quarks are labeled each by a color index ($a,\ b,\ c$ resp.), a flavor index ($i,\ j,\ k$ resp.) and, in the covariant version of the model, by a Dirac index ($\alpha,\ \beta,\ \gamma$ resp.) and a four-momentum ($p_1,\ p_2,\ p_3$ resp., with the constraint $p_1 + p_2 + p_3 = P$, P being the proton four-momentum). The complete proton wave function should be antisymmetric under the exchange of any two quarks. The proton being a color singlet, its (normalized) three-quark color wave function is given by the completely antisymmetric function $\varepsilon_{abc}/\sqrt{6}$. The remaining part of the proton wave function representing the distribution of spins, flavors and momenta of quarks should then be symmetric when exchanging any two quarks. It will be denoted by $\varphi(I, J, K)$, where I stands for the triplet $\{i, \alpha, p_1\}$, J for $\{j, \beta, p_2\}$, K for $\{k, \gamma, p_3\}$.

The infinitesimal generators of Lorentz transformations acting on $\varphi(I, J, K)$ are given by :

$$J_{\mu\nu} = \mathrm{S}_{\mu\nu} + i\sum_{r=1}^{r=3}\left(p_{r\mu}\frac{\partial}{\partial p_r{}^{\nu}} - p_{r\nu}\frac{\partial}{\partial p_r{}^{\mu}}\right) \tag{2.1}$$

where $\mathrm{S}_{\mu\nu}$ are the generators of the Lorentz spin group acting on the space E defined in the Introduction :

$$\mathrm{S}_{\mu\nu} = \sigma_{\mu\nu} \otimes 1_4 \otimes 1_4 + 1_4 \otimes \sigma_{\mu\nu} \otimes 1_4 + 1_4 \otimes 1_4 \otimes \sigma_{\mu\nu} \tag{2.2}$$

1_4 being the 4×4 unit matrix, and $\sigma_{\mu\nu} = i[\gamma_\mu, \gamma_\nu]/4$.

Defining relative four-momenta as :

$$p_\xi = \frac{1}{2}(p_1 - p_2) \tag{2.3a}$$

and :

$$p_\eta = \frac{1}{3}\,(p_1 + p_2 - 2p_3) \tag{2.3b}$$

we may rewrite Eq.(2.1) in the form :

$$J_{\mu\nu} = \mathcal{L}_{\mu\nu} + \mathcal{S}_{\mu\nu} \tag{2.4a}$$

where $\mathcal{L}_{\mu\nu}$ is the overall orbital angular momentum of the proton :

$$\mathcal{L}_{\mu\nu} = i\left(P_\mu \frac{\partial}{\partial P^\nu} - P_\nu \frac{\partial}{\partial P^\mu}\right) \tag{2.4b}$$

while $\mathcal{S}_{\mu\nu}$ is its spin angular momentum, including the spin part proper, $\mathrm{S}_{\mu\nu}$, and a relative orbital angular momentum part, $\mathrm{L}_{\mu\nu}$:

$$\mathcal{S}_{\mu\nu} = \mathrm{S}_{\mu\nu} + \mathrm{L}_{\mu\nu} \tag{2.4c}$$

where

$$\mathrm{L}_{\mu\nu} = \mathrm{L}^{(\xi)}_{\mu\nu} + \mathrm{L}^{(\eta)}_{\mu\nu} \tag{2.4d}$$

with

$$\mathrm{L}^{(\xi)}_{\mu\nu} = i\left(p_{\xi_\mu} \frac{\partial}{\partial {p_\xi}^\nu} - p_{\xi_\nu} \frac{\partial}{\partial {p_\xi}^\mu}\right) \tag{2.4e}$$

$$\mathrm{L}^{(\eta)}_{\mu\nu} = i\left(p_{\eta_\mu} \frac{\partial}{\partial {p_\eta}^\nu} - p_{\eta_\nu} \frac{\partial}{\partial {p_\eta}^\mu}\right) \tag{2.4f}$$

The spin of the proton is defined in the usual way through the Pauli-Lubanski operator :

$$\mathcal{W}_\mu = \frac{1}{2}\varepsilon_{\mu\nu\rho\sigma} P^\nu J^{\rho\sigma} \equiv \frac{1}{2}\varepsilon_{\mu\nu\rho\sigma} P^\nu \mathcal{S}^{\rho\sigma} \tag{2.5a}$$

the square of which characterizes the magnitude of the spin :

$$\mathcal{W}^2 \equiv \mathcal{W}_\mu \mathcal{W}^\mu = -M^2 J(J+1) \tag{2.5b}$$

Being a "low-lying" particle, the proton is usually considered as a S state as regards relative orbital angular momentum, which means that :

$$\frac{1}{2}\varepsilon_{\mu\nu\rho\sigma} P^\nu \mathrm{L}^{\rho\sigma} \varphi(I,J,K) = 0 \tag{2.6}$$

Consequently, $\varphi(I,J,K)$ should depend on p_ξ and p_η only through the invariant combinations $P.p_\xi$, $P.p_\eta$, p_ξ^2, p_η^2, and $p_\xi.p_\eta$.

Let us denote by $\{\psi^{(n)}_{\alpha\beta\gamma}(P)\ ,\ n = 1, ..., 64\}$ the 64-dimensional basis of E including all the third-rank tensor products constructed from the previously defined Dirac spinors $U(P)$ and $V(P) = \gamma_5 U(P)$. For example, one has $\psi^{(1)}_{\alpha\beta\gamma}(P) =$

$U^{\uparrow}_{\alpha}(P)U^{\uparrow}_{\beta}(P)U^{\uparrow}_{\gamma}(P)$, $\psi^{(8)}_{\alpha\beta\gamma}(P) = U^{\downarrow}_{\alpha}(P)U^{\downarrow}_{\beta}(P)U^{\downarrow}_{\gamma}(P)$, $\psi^{(9)}_{\alpha\beta\gamma}(P) = V^{\uparrow}_{\alpha}(P)U^{\uparrow}_{\beta}(P)U^{\uparrow}_{\gamma}(P)$ and so on.

We may then expand the wave function as[2] :

$$\varphi(I, J, K) = \sum_{n=1}^{n=64} \mathcal{F}^{(n)}_{ijk}(P.p_\xi, P.p_\eta, p^2_\xi, p^2_\eta, p_\xi.p_\eta, P^2)\psi^{(n)}_{\alpha\beta\gamma}(P) \tag{2.7}$$

where, by virtue of Eq.(2.6), the $\mathcal{F}^{(n)}$ are Lorentz-invariant components. From (2.6), we get in addition, using simplified notations :

$$J_{\mu\nu}\varphi = \sum \mathcal{F}^{(n)} \left(\mathrm{S}_{\mu\nu} + \mathcal{L}_{\mu\nu}\right) \psi^{(n)} \tag{2.8a}$$

and

$$\mathcal{W}^2\varphi = \sum \mathcal{F}^{(n)}\mathcal{W}^2_{(\mathrm{S})}\psi^{(n)} \tag{2.8b}$$

where $\mathcal{W}^2_{(\mathrm{S})}$ is the square of the Pauli-Lubanski operator $\mathcal{W}_{(\mathrm{S})\mu}$ associated with $\mathrm{S}_{\mu\nu}$, which has the same form as (2.5a), $J_{\mu\nu}$ being replaced by $\mathrm{S}_{\mu\nu}$.

These last two equations exhibit the obvious fact that, under the assumption that the proton is an S state, the covariance properties of its wave function are entirely determined by the effective combination of the $\psi^{(n)}$ in the expansion (2.7). In particular, we may now consider $\mathrm{S}_{\mu\nu}$ as the effective spin angular momentum of the proton.

The two Casimir operators that are invariant under the Lorentz spin group are given by :

$$C_1 = \frac{1}{4}\mathrm{S}_{\mu\nu}\mathrm{S}^{\mu\nu} = \frac{1}{2}\left(\vec{\mathrm{S}}^2 - \vec{\mathrm{N}}^2\right) \tag{2.9a}$$

and :

$$C_2 = -\frac{i}{8}\varepsilon_{\mu\nu\rho\sigma}\mathrm{S}^{\mu\nu}\mathrm{S}^{\rho\sigma} = -i\vec{\mathrm{S}}.\vec{\mathrm{N}} \tag{2.9b}$$

where $\vec{\mathrm{S}} = \{\mathrm{S}^1 = \mathrm{S}^{23},\ \mathrm{S}^2 = \mathrm{S}^{31},\ \mathrm{S}^3 = \mathrm{S}^{12}\}$ and $\vec{\mathrm{N}} = \{\mathrm{N}^1,\ \mathrm{N}^2,\ \mathrm{N}^3\}$ with $\mathrm{N}^k = \mathrm{S}^{0k}$ are the spin and boost operators respectively. As in the general theory of Lorentz-group representations [**9**], the irreducible representations of the Lorentz spin group in E will be labeled by the so-called associated quantum numbers j_1 and j_2 which define the eigenvalues of the squares of the following associated spin operators (which are commuting with each other) :

$$\vec{J_1} = \frac{1}{2}\left(\vec{\mathrm{S}} + i\vec{\mathrm{N}}\right) \quad , \quad \vec{J_2} = \frac{1}{2}\left(\vec{\mathrm{S}} - i\vec{\mathrm{N}}\right) \tag{2.10a}$$

through the equations :

[2] Apart from an overall (Lorentz-invariant) factor $\delta(P - p_1 - p_2 - p_3)$ expressing the constraint on the total four-momentum.

$$\vec{J}_1^{\,2} = j_1(j_1+1) \quad , \quad \vec{J}_2^{\,2} = j_2(j_2+1) \tag{2.10b}$$

One then gets :

$$C_1 = \vec{J}_1^{\,2} + \vec{J}_2^{\,2} \quad , \quad C_2 = \vec{J}_1^{\,2} - \vec{J}_2^{\,2} \tag{2.11}$$

It is worth mentioning here that, while the operator $\mathcal{W}^2$ is always commuting with all the generators (2.1) and, in this respect, is Lorentz-invariant, in general it does not commute separately with each of the two pieces (2.4b-c). Thus, generally speaking, $\mathcal{W}^2$ is not invariant under the Lorentz spin group. More specifically, in the present case where $\mathcal{W}^2 = \mathcal{W}^2_{(\mathrm{S})} \equiv -M^2\vec{\mathrm{S}}^2$ (in the proton rest-frame), one has :

$$\left[\, \vec{\mathrm{S}}^2, \vec{\mathrm{S}} \,\right] = \vec{0} \quad , \quad \left[\, \vec{\mathrm{S}}^2, \vec{\mathrm{N}} \,\right] = -4\vec{J}_1 \wedge \vec{J}_2 \tag{2.12}$$

which shows that, in a given representation $D(j_1, j_2)$, the boost operator is responsible for transitions between the various subspaces $D(J)$ entering into the decomposition (1.7). The only exceptions for which the spin is also invariant under the Lorentz spin group are the representations $D(J,0)$ and $D(0,J)$ corresponding to a single value of the spin, $J = j_1$ ($j_2 = 0$) and $J = j_2$ ($j_1 = 0$) respectively. These are just the minimal representations frequently considered in a general classification scheme of particles of any spin, according to irreducible representations of the full Poincaré group [**12**].

Let us now display the explicit matrix form of the various above-defined operators. First, we have :

$$\vec{\mathrm{S}} = \frac{1}{2}\left(\vec{\sigma}\otimes 1_4 \otimes 1_4 + 1_4 \otimes \vec{\sigma} \otimes 1_4 + 1_4 \otimes 1_4 \otimes \vec{\sigma}\right) \tag{2.13a}$$

and :

$$\vec{\mathrm{N}} = \frac{i}{2}\left(\vec{\sigma}\gamma_5 \otimes 1_4 \otimes 1_4 + 1_4 \otimes \vec{\sigma}\gamma_5 \otimes 1_4 + 1_4 \otimes 1_4 \otimes \vec{\sigma}\gamma_5\right) \tag{2.13b}$$

where $\vec{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$ stands for the usual Pauli matrices and where it is understood that :

$$\vec{\sigma} \equiv \vec{\sigma} 1_4 \equiv \begin{pmatrix} \vec{\sigma} & 0 \\ 0 & \vec{\sigma} \end{pmatrix} \quad , \quad \vec{\sigma}\gamma_5 \equiv \begin{pmatrix} 0 & \vec{\sigma} \\ \vec{\sigma} & 0 \end{pmatrix}$$

Then :

$$\vec{\mathrm{S}}^2 = \frac{3}{4}\left(3 + 2\,\mathcal{U}\right) \quad , \quad \vec{\mathrm{N}}^2 = -\frac{9}{4} + \frac{3}{2}\mathcal{M}_5 \tag{2.14a}$$

with :

$$\mathcal{U} = \frac{1}{3}\left(\vec{\sigma}.\otimes\vec{\sigma}\otimes 1_4 + \vec{\sigma}.\otimes 1_4 \otimes \vec{\sigma} + 1_4 \otimes \vec{\sigma}.\otimes \vec{\sigma}\right) \tag{2.14b}$$

$$\mathcal{M}_5 = -\frac{1}{3}\left(\vec{\sigma}\gamma_5.\otimes\vec{\sigma}\gamma_5\otimes 1_4 + \vec{\sigma}\gamma_5.\otimes 1_4 \otimes \vec{\sigma}\gamma_5 + 1_4 \otimes \vec{\sigma}\gamma_5.\otimes \vec{\sigma}\gamma_5\right) \tag{2.14c}$$

making use of the notation : $\vec{\sigma}.\otimes\vec{\sigma}\equiv\sum_i \sigma_i\otimes\sigma_i$. We thus get :

$$C_1=\frac{3}{4}\left(3+\mathcal{U}-\mathcal{M}_5\right) \tag{2.14d}$$

and, defining :

$$\Gamma_5=\gamma_5\otimes\gamma_5\otimes\gamma_5 \tag{2.15a}$$

$$\Lambda=\gamma_5\otimes 1_4\otimes 1_4+1_4\otimes\gamma_5\otimes 1_4+1_4\otimes 1_4\otimes\gamma_5 \tag{2.15b}$$

we obtain :

$$C_2=\frac{3}{4}\left[\left(\,1+\mathcal{U}\right)\Lambda+\Gamma_5\mathcal{M}_5\,\right] \tag{2.15c}$$

The above-defined operators $\mathcal{U},\ \mathcal{M}_5,\ \Lambda$ and Γ_5 are not independent. In fact, the operators

$$u_1=\vec{\sigma}.\otimes\vec{\sigma}\otimes 1_4\quad,\quad u_2=1_4\otimes\vec{\sigma}.\otimes\vec{\sigma}\quad,\quad u_3=\vec{\sigma}.\otimes 1_4\otimes\vec{\sigma} \tag{2.16a}$$

satisfy the identities :

$$u_i^2=3-2u_i\quad,\quad u_iu_j+u_ju_i=2u_k\ (i\neq j\neq k\neq i) \tag{2.16b}$$

$$u_i(1+\mathcal{U})=(1+\mathcal{U})u_i=1+\mathcal{U} \tag{2.16c}$$

from which it is easy to show that :

$$\mathcal{U}=\frac{3}{2}\left(1-\frac{2}{3}\mathcal{M}_5-\mathcal{M}_5^2\right) \tag{2.17a}$$

$$\left(1+\mathcal{U}\right)\mathcal{M}_5=-\frac{1}{3}\left(1+\mathcal{U}\right)\Gamma_5\Lambda \tag{2.17b}$$

Moreover, one has :

$$\left(\mathcal{U}+\mathcal{M}_5\right)\left(1+\Gamma_5\Lambda\right)=0 \tag{2.17c}$$

$$\Gamma_5=\frac{1}{2}\left(\Lambda-3\Lambda^{-1}\right) \tag{2.17d}$$

One should take Eqs.(2.17) into account, especially when investigating the eigenvalue spectra of the operators involved. From (2.16) one readily gets the equation $\mathcal{U}^2=1$ showing that the eigenvalues of $\mathcal{U}$ are $u=\pm1$. Setting $\vec{\mathrm{S}}^2=J(J+1)$ in (2.14a), J being a spin value, one thus obtains the two expected spin values $J=1/2$ and $J=3/2$ corresponding to $u=-1$ and $u=+1$ respectively. The eigenvalues of $\mathcal{M}_5$ are easily derived from (2.17a) : they are $m_5 = +1$ and $m_5 = -5/3$ for $u = -1$, $m_5 = 1/3$ and $m_5 = -1$ for $u = 1$. On the other hand, since $\Gamma_5^2 = 1$, the eigenvalues of Γ_5 are $\gamma = \pm 1$ and, from Eq. (2.17d), those of Λ are found to be $\lambda = -1$ and $\lambda = 3$ for $\gamma = 1$, and $\lambda = 1$ and $\lambda = -3$ for $\gamma = -1$, with the restriction that, because of Eq. (2.17b-c), they must satisfy the constraints : $\gamma\lambda = -3m_5$ if $u = 1$ and $\gamma\lambda = -1$ if $u + m_5 \neq 0$. The irreducible representations of the Lorentz spin group in E may now be listed as follows.

- A) Case $u = -1$, $m_5 = +1$ ($J = 1/2$).

  In that case, since $1 + u = 0$ and $u + m_5 = 0$, the eigenvalues of $\Gamma_5\Lambda$ are uncorrelated with those of $\mathcal{U}$ and $\mathcal{M}_5$, and the eigenvalues of C_1 and C_2 do not depend on λ. We thus find the following possibilities :

  a) $\gamma = 1$.

  In that case $C_1 = C_2 = 3/4$, hence $j_1 = 1/2$, $j_2 = 0$. This corresponds to a representation $D(\frac{1}{2}, 0)$ which is obtained from either i) $\lambda = -1$ or ii) $\lambda = 3$.

  b) $\gamma = -1$.

  Here we have $C_1 = -C_2 = 3/4$ and $j_1 = 0$, $j_2 = 1/2$. This gives rise to a representation $D(0, \frac{1}{2})$, obtained from either i) $\lambda = 1$ or ii) $\lambda = -3$.

- B) Case $u = -1$, $m_5 = -5/3$ ($J = 1/2$).

  Here the constraint $\gamma\lambda = -1$ is effective, so that only two cases are possible :

  a) $\gamma = 1$, $\lambda = -1$.

  In that case $C_1 = 11/4$, $C_2 = -5/4$ and $j_1 = 1/2$, $j_2 = 1$. This corresponds to the subspace of a representation $D(\frac{1}{2}, 1)$ associated with the spin value $J = 1/2$.

  b) $\gamma = -1$, $\lambda = 1$.

  Hence $C_1 = 11/4$, $C_2 = 5/4$ and $j_1 = 1$, $j_2 = 1/2$. We here get the subspace of $D(1, \frac{1}{2})$ associated with $J = 1/2$.

- C) Case $u = 1$, $m_5 = 1/3$ ($J = 3/2$).

  Here we must have $\gamma\lambda = -3m_5 = -1$ (see Eq.(2.17)) and we obtain the subspaces of $D(\frac{1}{2}, 1)$ and $D(1, \frac{1}{2})$, which are associated with the spin value $J = 3/2$ and correspond respectively to each of the following cases :

  a) $\gamma = 1$, $\lambda = -1$, hence $C_1 = 11/4$, $C_2 = -5/4$ and $j_1 = 1/2$, $j_2 = 1$.

  b) $\gamma = -1$, $\lambda = 1$, hence $C_1 = 11/4$, $C_2 = 5/4$ and $j_1 = 1$, $j_2 = 1/2$.

- D) Case $u = 1$, $m_5 = -1$ ($J = 3/2$).

  This time we have $\gamma\lambda = -3m_5 = 3$ and we get the two representations $D(\frac{3}{2}, 0)$ and $D(0, \frac{3}{2})$ corresponding respectively to each of the following cases :

  a) $\gamma = 1$, $\lambda = 3$, hence $C_1 = C_2 = 15/4$ and $j_1 = 3/2$, $j_2 = 0$.

  b) $\gamma = -1$, $\lambda = -3$, hence $C_1 = -C_2 = 15/4$ and $j_1 = 0$, $j_2 = 3/2$.

As a result, one may characterize representations like $D(j_1, j_2) \oplus D(j_2, j_1)$ (which are representations of the full Lorentz spin group in E)[3] only by eigenvalues of $\mathcal{M}_5$ and (possibly) $\mathcal{U}$. This is summarized in Table I. What are the respective dimensions of these representations? They are readily derived from Eq.(1.6) but one may verify them by computing the trace of the projection operator $\mathcal{Q}_{m_5}$ of each representation. We have :

$$\mathcal{Q}_1 = \frac{3}{16}\,(1 - \mathcal{U})\left(\mathcal{M}_5 + \frac{5}{3}\right) \tag{2.18a}$$

for $E_1 = D(\frac{1}{2}, 0) \oplus D(0, \frac{1}{2})$,

$$\mathcal{Q}_{-5/3} = \frac{3}{16}\,(1 - \mathcal{U})\,(1 - \mathcal{M}_5) \tag{2.18b}$$

for the subspace $E_{-5/3}$ of $D(\frac{1}{2}, 1) \oplus D(1, \frac{1}{2})$ corresponding to $J = 1/2$,

$$\mathcal{Q}_{1/3} = \frac{3}{8}\,(1 + \mathcal{U})\,(1 + \mathcal{M}_5) \tag{2.18c}$$

for the subspace $E_{1/3}$ of $D(\frac{1}{2}, 1) \oplus D(1, \frac{1}{2})$ corresponding to $J = 3/2$,

$$\mathcal{Q}_{-1} = \frac{3}{8}\,(1 + \mathcal{U})\left(\frac{1}{3} - \mathcal{M}_5\right) \tag{2.18d}$$

for $E_{-1} = D(\frac{3}{2}, 0) \oplus D(0, \frac{3}{2})$.

Using the simple relation $\mathrm{Tr}(A \otimes B \otimes C) = (\mathrm{Tr}A)(\mathrm{Tr}B)(\mathrm{Tr}C)$ we get $\mathrm{Tr}\ \mathcal{U} = \mathrm{Tr}\ \mathcal{M}_5 = \mathrm{Tr}\ \mathcal{U}\mathcal{M}_5 = 0$ and hence :

$$\begin{aligned} \mathrm{dim}E_1 &= \mathrm{Tr}\ \mathcal{Q}_1 = 20 \\ \mathrm{dim}(E_{-5/3} \oplus E_{1/3}) &= \mathrm{Tr}(\mathcal{Q}_{-5/3} + \mathcal{Q}_{1/3}) = 36 \\ \mathrm{dim}E_{-1} &= \mathrm{Tr}\ \mathcal{Q}_{-1} = 8 \end{aligned} \tag{2.19}$$

[3] It is understood here that these representations are not irreducible in general, in the sense that they may be the direct sum of several irreducible representations $D(j_1, j_2) \oplus D(j_2, j_1)$ each having the required *minimal* dimension $2(2j_1 + 1)(2j_2 + 1)$. For simplicity, we use the same notation for this kind of representation in all cases.

Since irreducible representations $D(\frac{1}{2},0)\oplus D(0,\frac{1}{2})$, $D(\frac{1}{2},1)\oplus D(1,\frac{1}{2})$ and $D(\frac{3}{2},0)\oplus D(0,\frac{3}{2})$ have respective dimensions 4, 12 and 8, each of them appears then in E respectively 5 times, 3 times and once, in agreement with Eq.(1.6). On the other hand, since $\dim E_{-5/3}=12$ and $\dim E_{1/3}=24$, the two subspaces $E_1\oplus E_{-5/3}$ and $E_{1/3}\oplus E_{-1}$ which correspond respectively to $J=1/2$ and $J=3/2$ have dimension 32 each. In the next section we give a 64-dimensional basis of eigenspinors of $\mathcal{M}_5$ and $\mathcal{U}$, which we classify according to the reduction rule (1.6).

3. BASIS OF EIGENSPINORS

In constructing the three-quark proton wave function, one usually classifies three-quark spinor forms of definite spin according to irreducible representations of the permutation group S_3. Afterwards, one couples spinor forms of a given symmetry under S_3 with three-quark flavor wave functions displaying the same symmetry, in order to obtain a fully symmetric spin-flavor proton wave function [**13**].

In terms of Young tableaux, the reduction of the space E with respect to S_3 reads [**13, 14**] :

$$\begin{array}{|c|}\hline \alpha \\ \hline\end{array} \otimes \begin{array}{|c|}\hline \beta \\ \hline\end{array} \otimes \begin{array}{|c|}\hline \gamma \\ \hline\end{array} = \begin{array}{|c|c|c|}\hline \alpha & \beta & \gamma \\ \hline\end{array} \oplus \begin{array}{|c|}\hline \alpha \\ \hline \beta \\ \hline \gamma \\ \hline\end{array} \oplus \begin{array}{|c|c|}\hline \alpha & \gamma \\ \hline \beta \\ \cline{1-1}\end{array} \oplus \begin{array}{|c|c|}\hline \alpha & \beta \\ \hline \gamma \\ \cline{1-1}\end{array} \tag{3.1}$$

which means that any third-rank spinor $\varphi_{\alpha\beta\gamma}$ may be viewed as the sum of four independent third-rank spinors, namely i) a completely symmetric one (belonging to an irreducible subspace Y_S), ii) a second one which is completely antisymmetric (irreducible subspace Y_A), iii) a third one which has only antisymmetry under permutation of the two first indices α and β (subspace Y_{12}) and, finally, iv) a fourth one which is antisymmetric under the permutation of α and γ (subspace Y_{13}). The last two species are usually arranged in a mixed symmetry doublet (irreducible subspace $Y_D=Y_{12}\oplus Y_{13}$) with one component antisymmetric under the permutation of α and β (subspace $Y_{M(A)}$) while its second component is symmetric under that permutation (subspace $Y_{M(S)}$).

Each above-defined subspace Y is associated with a projection operator $\mathcal{Y}$ (a so-called Young idempotent). For Y_S, Y_A, Y_{12}, Y_{13}, $Y_{M(A)}$, $Y_{M(S)}$ and Y_D respectively, it is :

$$\mathcal{Y}_S=\frac{1}{6}\left(1+\mathcal{P}_{12}+\mathcal{P}_{13}+\mathcal{P}_{23}+\mathcal{P}_{12}\mathcal{P}_{13}+\mathcal{P}_{13}\mathcal{P}_{12}\right) \tag{3.2a}$$

$$\mathcal{Y}_A=\frac{1}{6}\left(1-\mathcal{P}_{12}-\mathcal{P}_{13}-\mathcal{P}_{23}+\mathcal{P}_{12}\mathcal{P}_{13}+\mathcal{P}_{13}\mathcal{P}_{12}\right) \tag{3.2b}$$

$$\mathcal{Y}_{12}=\frac{1}{3}\left(1+\mathcal{P}_{13}\right)\left(1-\mathcal{P}_{12}\right) \quad , \quad \mathcal{Y}_{13}=\frac{1}{3}\left(1+\mathcal{P}_{12}\right)\left(1-\mathcal{P}_{13}\right) \tag{3.2c}$$

$$\mathcal{Y}_{M(A)}=\frac{1}{6}\left(1-\mathcal{P}_{12}\right)\left(1+\mathcal{P}_{13}\right)\left(1-\mathcal{P}_{12}\right) \tag{3.2d}$$

$$\mathcal{Y}_{M(S)} = \frac{1}{6}\,(1+\mathcal{P}_{12})\,(1-\mathcal{P}_{13})\,(1+\mathcal{P}_{12}) \tag{3.2e}$$

$$\mathcal{Y}_D = \mathcal{Y}_{12} + \mathcal{Y}_{13} = \mathcal{Y}_{M(A)} + \mathcal{Y}_{M(S)} \tag{3.2f}$$

where, for instance, $\mathcal{P}_{12}$ is the permutation operator that interchanges the two first indices α and β :

$$(\mathcal{P}_{12})_{\alpha\beta\gamma}^{\alpha'\beta'\gamma'} = \delta_\beta^{\alpha'}\delta_\alpha^{\beta'}\delta_\gamma^{\gamma'} \tag{3.3}$$

Remember the obvious fact that the reduction (3.1) is also Lorentz-invariant. To get an idea on how the irreducible representations of S_3 and those of the Lorentz spin group are overlapping in E, we may compute the traces $\mathrm{Tr}\,(\mathcal{Y}\ \mathcal{Q}_{m_5})$, which define the dimensions of the intersections between subspaces Y and subspaces E_{m_5} defined in Sec.2 :

$$\dim(Y \cap E_{m_5}) = \mathrm{Tr}\ \mathcal{Y}\mathcal{Q}_{m_5} \tag{3.4}$$

Using relations :

$$\mathrm{Tr}(\mathcal{P}_{12} A \otimes B \otimes C) = (\mathrm{Tr} AB)\,\mathrm{Tr} C \tag{3.5a}$$

$$\mathrm{Tr}(\mathcal{P}_{12}\mathcal{P}_{13} A \otimes B \otimes C) = \mathrm{Tr} ABC \tag{3.5b}$$

and similar ones obtained by successive permutations of indices, we get :

$$\mathrm{Tr}\ \mathcal{P}_{rs}\,\mathcal{U} = -\mathrm{Tr}\ \mathcal{P}_{rs}\ \mathcal{M}_5 = 3\mathrm{Tr}\ \mathcal{P}_{rs}\,\mathcal{U}\mathcal{M}_5 = 16 \tag{3.6a}$$

$$\mathrm{Tr}\ \mathcal{P}_{12}\mathcal{P}_{13}\,\mathcal{U} = -\mathrm{Tr}\ \mathcal{P}_{12}\mathcal{P}_{13}\ \mathcal{M}_5 = -3\mathrm{Tr}\ \mathcal{P}_{12}\mathcal{P}_{13}\,\mathcal{U}\mathcal{M}_5 = 12 \tag{3.6b}$$

$$\mathrm{Tr}\ \mathcal{P}_{rs} = 4\mathrm{Tr}\ \mathcal{P}_{12}\mathcal{P}_{13} = 16 \tag{3.6c}$$

from which the various dimensions (3.4) are easily derived. They are listed in Table II as well as the dim$Y$ which, from theory, are given by [**13**, **14**] :

$$\mathrm{dim}Y_A = \frac{N!}{p!(N-p)!} = 4$$

$$\mathrm{dim}Y_S = \frac{(N+p-1)!}{p!(N-1)!} = 20$$

$$\mathrm{dim}Y_{M(A)} = \mathrm{dim}Y_{M(S)} = \mathrm{dim}Y_D/2 = N(N^2-1)/3 = 20$$

since we are here dealing with third-rank spinors ($p = 3$) constructed from Dirac spinors ($N = 4$).

A given spinor form corresponding to definite values of parity, spin and third-component of spin S^3 may serve, in fact, to generate three other spinor forms having opposite parity or opposite value of S^3, or both, by application of a parity-reversing operation (defined below) or/and spin-reversing operation. It is thus sufficient to consider only generic forms. We may then classify the basis of each representation $D(j_1, j_2) \oplus D(j_2, j_1)$ by means of generic irreducible representations of S_3, namely generic doublets $\mathcal{D}$ (each leading to 8 spinor forms), generic antisymmetric singlets $\mathcal{A}$ (giving each 4 spinor forms) and generic symmetric singlets $\mathcal{S}$ (4 spinor forms for each). Taking the results of Table II into account, we find that :

- $D(\frac{1}{2}, 0) \oplus D(0, \frac{1}{2})$ consists of two $\mathcal{D}$ and the only $\mathcal{A}$ appearing in the reduction.
- $D(\frac{1}{2}, 1) \oplus D(1, \frac{1}{2})$ is made up of :

  a) a $\mathcal{D}$ and a $\mathcal{S}$ corresponding to the spin value $J = 1/2$,

  b) two $\mathcal{D}$ and two $\mathcal{S}$ corresponding to $J = 3/2$ (one $\mathcal{D}$ and one $\mathcal{S}$ for $\mathrm{S}^3 = +1/2$, one $\mathcal{D}$ and one $\mathcal{S}$ for $\mathrm{S}^3 = +3/2$).
- $D(\frac{3}{2}, 0) \oplus D(0, \frac{3}{2})$ is resolved into two $\mathcal{S}$ (one for $\mathrm{S}^3 = +1/2$, the other for $\mathrm{S}^3 = +3/2$).

The 64-dimensional basis of eigenspinors of $\mathcal{M}_5$ and $\mathcal{U}$ will be formed from the third-rank spinors $\psi^{(n)}$ defined in Sec.2 (Eq.(2.7)) and its explicit construction will be carried out as follows.

For the sake of simplicity, it is convenient to search for such eigenspinors first in the proton rest frame where the spin of the proton is simply represented by the operators $\vec{\mathrm{S}}$ of Eq.(2.13a)[4] and the parity operator is also simply represented by the matrix :

$$\Gamma_0 = \gamma_0 \otimes \gamma_0 \otimes \gamma_0 \tag{3.7}$$

Afterwards, eigenspinors associated to a given current value P of the proton four-momentum will be derived in an obvious way by a Lorentz transformation which amounts to replace the 4-spinors U and V defined in the proton rest frame simply by $U(P)$ and $V(P)$ respectively.

Next, we select from the $\psi^{(n)}$ the generic forms UUU, VVU, VUV and UVV which have positive parity[5]. The remaining forms VVV, UUV, UVU and VUU lead to negative-parity wave functions and may be obtained by applying the parity-reversing operator Γ_5 (Eq.(2.15a)) to the preceding ones. They may be associated with antibaryon wave functions. Also, to save a lot of typing, tensor products of Dirac spinors like $\psi_1 \otimes \psi_2$ and $\psi_1 \otimes \psi_2 \otimes \psi_3$ will be subsequently denoted simply by $\psi_1\psi_2$ and $\psi_1\psi_2\psi_3$.

[4]In a frame where the proton has four-momentum P, its spin operators, which generate the so-called little group of P, are defined by $\mathrm{S}^i(P) = -\mathcal{W}_\mu n_i^\mu(P)/M$ with $i = 1, 2, 3$, the $n_i^\mu(P)$ being three spacelike 4-vectors orthogonal to P (see P. Moussa and R. Stora, Ref. [**12**]).

[5]Let us here remark that in the proton rest frame the Dirac equations for U and $V = \gamma_5 U$ amount to $\gamma_0 U = U$ and $\gamma_0 V = -V$.

From the relations :

$$\sum_{k=1}^{3}\left(\sigma_k U^{\uparrow,\downarrow}\right)\left(\sigma_k U^{\uparrow,\downarrow}\right) \equiv \left(\vec{\sigma} U^{\uparrow,\downarrow}\right).\left(\vec{\sigma} U^{\uparrow,\downarrow}\right) = U^{\uparrow,\downarrow}U^{\uparrow,\downarrow} \tag{3.8a}$$

$$\left(\vec{\sigma} U^{\uparrow,\downarrow}\right).\left(\vec{\sigma} U^{\downarrow,\uparrow}\right) = 2U^{\downarrow,\uparrow}U^{\uparrow,\downarrow} - U^{\uparrow,\downarrow}U^{\downarrow,\uparrow} \tag{3.8b}$$

which are valid in the proton rest frame, follows the well-known result that the combinations $(U^\uparrow U^\downarrow - U^\downarrow U^\uparrow)U^\uparrow$ and $(U^\uparrow U^\downarrow + U^\downarrow U^\uparrow)U^\uparrow - 2U^\uparrow U^\uparrow U^\downarrow$ are spin 1/2 wave functions, being eigenspinors of $\mathcal{U}$ with eigenvalue $u = -1$, and that $U^\uparrow U^\uparrow U^\downarrow + U^\uparrow U^\downarrow U^\uparrow + U^\downarrow U^\uparrow U^\uparrow$ and $U^\uparrow U^\uparrow U^\uparrow$ are spin 3/2 wave functions, as they are eigenspinors of $\mathcal{U}$ with $u = +1$. Applying the parity-conserving operators $\gamma_5 \otimes \gamma_5 \otimes 1_4$, $\gamma_5 \otimes 1_4 \otimes \gamma_5$ and $1_4 \otimes \gamma_5 \otimes \gamma_5$ successively to those spinor forms it is then straightforward to show that the subspace $\mathcal{E}_{1/2}$ of spin 1/2 wave functions is generated by the eight following basic third-rank spinors (of positive parity and corresponding to $\mathrm{S}^3 = +1/2$) :

$$v_1 = \tfrac{1}{\sqrt{2}}\left(U^\uparrow U^\downarrow - U^\downarrow U^\uparrow\right)U^\uparrow \tag{3.9a}$$

$$v_2 = \tfrac{1}{\sqrt{2}}\left(V^\uparrow V^\downarrow - V^\downarrow V^\uparrow\right)U^\uparrow \tag{3.9b}$$

$$v_3 = \tfrac{1}{\sqrt{2}}\left(V^\uparrow U^\downarrow - V^\downarrow U^\uparrow\right)V^\uparrow \tag{3.9c}$$

$$v_4 = \tfrac{1}{\sqrt{2}}\left(U^\uparrow V^\downarrow - U^\downarrow V^\uparrow\right)V^\uparrow \tag{3.9d}$$

$$v_5 = \tfrac{1}{\sqrt{6}}\left\{\left(U^\uparrow U^\downarrow + U^\downarrow U^\uparrow\right)U^\uparrow - 2U^\uparrow U^\uparrow U^\downarrow\right\} \tag{3.9e}$$

$$v_6 = \tfrac{1}{\sqrt{6}}\left\{\left(V^\uparrow V^\downarrow + V^\downarrow V^\uparrow\right)U^\uparrow - 2V^\uparrow V^\uparrow U^\downarrow\right\} \tag{3.9f}$$

$$v_7 = \tfrac{1}{\sqrt{6}}\left\{\left(V^\uparrow U^\downarrow + V^\downarrow U^\uparrow\right)V^\uparrow - 2V^\uparrow U^\uparrow V^\downarrow\right\} \tag{3.9g}$$

$$v_8 = \tfrac{1}{\sqrt{6}}\left\{\left(U^\uparrow V^\downarrow + U^\downarrow V^\uparrow\right)V^\uparrow - 2U^\uparrow V^\uparrow V^\downarrow\right\} \tag{3.9h}$$

while the eight generic third-rank spinors (of positive parity) for the subspace $\mathcal{E}_{3/2}$ of spin 3/2 wave functions are :

$$v'_1 = \tfrac{1}{\sqrt{3}}\left\{U^\uparrow U^\uparrow U^\downarrow + U^\uparrow U^\downarrow U^\uparrow + U^\downarrow U^\uparrow U^\uparrow\right\} \tag{3.10a}$$

$$v'_2 = \tfrac{1}{\sqrt{3}}\left\{V^\uparrow V^\uparrow U^\downarrow + V^\uparrow V^\downarrow U^\uparrow + V^\downarrow V^\uparrow U^\uparrow\right\} \tag{3.10b}$$

$$v'_3 = \tfrac{1}{\sqrt{3}}\left\{V^\uparrow U^\uparrow V^\downarrow + V^\uparrow U^\downarrow V^\uparrow + V^\downarrow U^\uparrow V^\uparrow\right\} \tag{3.10c}$$

$$v'_4 = \tfrac{1}{\sqrt{3}}\left\{U^\uparrow V^\uparrow V^\downarrow + U^\uparrow V^\downarrow V^\uparrow + U^\downarrow V^\uparrow V^\uparrow\right\} \tag{3.10d}$$

for $\mathrm{S}^3 = +1/2$ and :

$$v'_5 = U^\uparrow U^\uparrow U^\uparrow \tag{3.10e}$$

$$v'_6 = V^\uparrow V^\uparrow U^\uparrow \tag{3.10f}$$

$$v'_7 = V^\uparrow U^\uparrow V^\uparrow \tag{3.10g}$$

$$v'_8 = U^\uparrow V^\uparrow V^\uparrow \tag{3.10h}$$

for $S^3 = +3/2$. Here, all spinor forms have been normalized according to $\bar{\psi}\psi \equiv \psi^\dagger\Gamma_0\psi = 1$, assuming $\bar{U}U = -\bar{V}V = 1$. In addition, they are orthogonal to each other.

Let us concentrate particularly on spin 1/2 third-rank spinors and classify them into $\mathcal{D}$-, $\mathcal{A}$- and $\mathcal{S}$-representations of S_3. Among the v (Eq.(3.9)) only v_1 and v_2 are antisymmetric, and v_5 and v_6 symmetric, under the permutation $\mathcal{P}_{12}$. However, since :

$$\mathcal{P}_{12}\ v_3 = -v_4 \quad , \quad \mathcal{P}_{12}\ v_7 = v_8 \tag{3.11}$$

the four remaining forms v_3, v_4, v_7 and v_8 may be arranged into two antisymmetric combinations v_3+v_4 and v_7-v_8, and two symmetric ones v_3-v_4 and v_7+v_8. As seen above, the four antisymmetric forms v_1, v_2, v_3+v_4 and v_7-v_8 must fit together into the antisymmetric components (a) of three doublets and an antisymmetric singlet $\mathcal{A}$ while the symmetric components (s) of those doublets plus a symmetric singlet must result from the four symmetric forms v_5, v_6, v_3-v_4 and v_7+v_8. To see this, we first apply the projection operator $\mathcal{Y}_{M(A)}$ to the antisymmetric set. This yields :

$$\mathcal{Y}_{M(A)}\ v_1 = v_1 \tag{3.12a}$$
$$\mathcal{Y}_{M(A)}\ v_2 = \tfrac{2}{3}v_2 + \tfrac{1}{6}(v_3+v_4) - \tfrac{1}{2\sqrt{3}}(v_7-v_8) \tag{3.12b}$$
$$\mathcal{Y}_{M(A)}\ (v_3+v_4) = \tfrac{1}{3}v_2 + \tfrac{5}{6}(v_3+v_4) + \tfrac{1}{2\sqrt{3}}(v_7-v_8) \tag{3.12c}$$
$$\mathcal{Y}_{M(A)}\ (v_7-v_8) = -\tfrac{1}{\sqrt{3}}v_2 + \tfrac{1}{2\sqrt{3}}(v_3+v_4) + \tfrac{1}{2}(v_7-v_8) \tag{3.12d}$$

It follows that the combinations :

$$a_1 \equiv v_1 \tag{3.13a}$$
$$a_2 = \tfrac{1}{\sqrt{3}}(v_2+v_3+v_4) \tag{3.13b}$$
$$a_3 = \tfrac{1}{2}\left\{\tfrac{1}{\sqrt{3}}(-2v_2+v_3+v_4) + v_7 - v_8\right\} \tag{3.13c}$$

are eigenspinors of $\mathcal{Y}_{M(A)}$ with eigenvalue +1 : these are just the components (a) of the doublets we are looking for[6]. On the other hand, $\mathcal{Y}_{M(A)}$ has a single eigenspinor with eigenvalue 0 which is the expected antisymmetric singlet :

$$\mathcal{A} = \frac{1}{2}\left\{\frac{1}{\sqrt{3}}(-2v_2+v_3+v_4) - v_7 + v_8\right\} \tag{3.14}$$

Remember that under the permutation $\mathcal{P}_{13}$ a doublet $\mathcal{D} = \begin{pmatrix} a \\ s \end{pmatrix}$ transforms as :

$$\mathcal{D}' \equiv \mathcal{P}_{13}\mathcal{D} = \frac{1}{2}\begin{pmatrix} 1 & \sqrt{3} \\ \sqrt{3} & -1 \end{pmatrix}\mathcal{D} \tag{3.15}$$

[6] Of course, any of those three forms may as well be replaced by a linear combination of them.

provided its components are normalized to unity. Thus, the symmetric partners of the (a) in Eq.(3.13) may be derived directly from the relation :

$$s = \frac{1}{\sqrt{3}} \left(2\mathcal{P}_{13} a - a \right) \tag{3.16}$$

Proceeding in this way, we find :

$$s_1 \equiv v_5 \tag{3.17a}$$
$$s_2 = \tfrac{1}{\sqrt{3}} (v_6 + v_7 + v_8) \tag{3.17b}$$
$$s_3 = \tfrac{1}{2} \left\{ \tfrac{1}{\sqrt{3}} (2v_6 - v_7 - v_8) + v_3 - v_4 \right\} \tag{3.17c}$$

Finally, the symmetric singlet is to be found as the (single) linear combination of v_6, $v_3 - v_4$ and $v_7 + v_8$ that satisfies the equation $\mathcal{Y}_{M(S)} \mathcal{S} = 0$. This yields :

$$\mathcal{S} = \frac{1}{2} \left\{ \frac{1}{\sqrt{3}} (2v_6 - v_7 - v_8) - v_3 + v_4 \right\} \tag{3.18}$$

From Table II we already know that the singlets $\mathcal{A}$ and $\mathcal{S}$ belong respectively to the subspaces E_1 and $E_{-5/3}$. To find out how the above-defined doublets are distributed among those two subspaces, we apply to them the operator $\mathcal{M}_5$ (Eq.(2.14c)). After some algebra, we get :

$$\mathcal{M}_5 \ \mathcal{D}_1 = \tfrac{1}{\sqrt{3}} \left(\mathcal{D}_2 - 2\mathcal{D}_3 \right) \tag{3.19a}$$
$$\mathcal{M}_5 \ \mathcal{D}_2 = \tfrac{1}{\sqrt{3}} \mathcal{D}_1 + \tfrac{2}{3} \left(\mathcal{D}_2 + 2\mathcal{D}_3 \right) \tag{3.19b}$$
$$\mathcal{M}_5 \ \mathcal{D}_3 = -\tfrac{2}{\sqrt{3}} \mathcal{D}_1 + \tfrac{1}{3} \left(2\mathcal{D}_2 - \mathcal{D}_3 \right) \tag{3.19c}$$

Hence, the doublets :

$$\mathrm{D}_1^{(1)} = \tfrac{1}{2} \left(\mathcal{D}_1 + \sqrt{3} \mathcal{D}_2 \right) \tag{3.20a}$$
$$\mathrm{D}_1^{(2)} = \tfrac{1}{2\sqrt{2}} \left(\sqrt{3} \mathcal{D}_1 - \mathcal{D}_2 - 2\mathcal{D}_3 \right) \tag{3.20b}$$
$$\mathrm{D}_{-5/3}^{(1)} = \tfrac{1}{2\sqrt{2}} \left(\sqrt{3} \mathcal{D}_1 - \mathcal{D}_2 + 2\mathcal{D}_3 \right) \tag{3.20c}$$

diagonalize $\mathcal{M}_5$ with eigenvalue $m_5 = +1$ for the first two and eigenvalue $m_5 = -5/3$ for the third one.

• We then conclude that $\mathrm{D}_1^{(1)}$, $\mathrm{D}_1^{(2)}$ and $\mathcal{A}$ on the one hand, and $\mathrm{D}_{-5/3}^{(1)}$ and $\mathcal{S}$ on the other hand, are providing the generic basis of, respectively, E_1 and $E_{-5/3}$.

The case of spin 3/2 third-rank spinors is easier to handle. From (3.10) we readily obtain the two symmetric singlets $\mathcal{S}'_1 = v'_1$ and $\mathcal{S}''_1 = v'_5$. On the other hand, only two forms antisymmetric under $\mathcal{P}_{12}$ may be constructed, namely $v'_3 - v'_4$ and

$v'_7 - v'_8$. These forms cannot mix since they are associated with different values of S^3. Hence, each of them must define the antisymmetric component of a doublet, the symmetric component of the latter being derived again from (3.16). We thus get the two expected doublets of $E_{1/3}$:

$$\mathrm{D}^{(1)}_{1/3} = \begin{pmatrix} a'_1 = \frac{1}{\sqrt{2}}(v'_3 - v'_4) \\ s'_1 = \frac{1}{\sqrt{6}}(v'_3 + v'_4 - 2v'_2) \end{pmatrix} \quad (\mathrm{S}^3 = +1/2) \tag{3.21a}$$

$$\mathrm{D}^{(2)}_{1/3} = \begin{pmatrix} a'_2 = \frac{1}{\sqrt{2}}(v'_7 - v'_8) \\ s'_2 = \frac{1}{\sqrt{6}}(v'_7 + v'_8 - 2v'_6) \end{pmatrix} \quad (\mathrm{S}^3 = +3/2) \tag{3.21b}$$

The remaining two symmetric singlets to be established should be constructed separately from the symmetric forms v'_2 and $v'_3 + v'_4$ on the one hand ($\mathrm{S}^3 = +1/2$) and v'_6 and $v'_7 + v'_8$ on the other hand ($\mathrm{S}^3 = +3/2$). Taking them orthogonal to s'_1 and s'_2 respectively, we find immediately :

$$\mathcal{S}'_2 = \frac{1}{\sqrt{3}}\,(v'_2 + v'_3 + v'_4) \tag{3.22a}$$

$$\mathcal{S}''_2 = \frac{1}{\sqrt{3}}\,(v'_6 + v'_7 + v'_8) \tag{3.22b}$$

Now we have $u = +1$ ($J = 3/2$) and from (2.17b) the operator $\mathcal{M}_5$ can conveniently be replaced by the more manageable one $-\Gamma_5\Lambda/3$ ($m_5 = -\gamma\lambda/3$). From :

$$\Gamma_5\Lambda\mathcal{S}'_1 = \Gamma_5\Lambda v'_1 = \sqrt{3}\mathcal{S}'_2 \tag{3.23a}$$

$$\Gamma_5\Lambda\mathcal{S}'_2 = 2\mathcal{S}'_2 + \sqrt{3}v'_1 \tag{3.23b}$$

$$\Gamma_5\Lambda\mathcal{S}''_1 = \Gamma_5\Lambda v'_5 = \sqrt{3}\mathcal{S}''_2 \tag{3.23c}$$

$$\Gamma_5\Lambda\mathcal{S}''_2 = 2\mathcal{S}''_2 + \sqrt{3}v'_5 \tag{3.23d}$$

it follows that :

$$\mathcal{S}^{(1)}_{1/3} = \frac{1}{2}\left(\sqrt{3}v'_1 - \mathcal{S}'_2\right) \quad (\mathrm{S}^3 = +1/2) \tag{3.24a}$$

$$\mathcal{S}^{(2)}_{1/3} = \frac{1}{2}\left(\sqrt{3}v'_5 - \mathcal{S}''_2\right) \quad (\mathrm{S}^3 = +3/2) \tag{3.24b}$$

are the symmetric singlets corresponding to $m_5 = 1/3$, and that :

$$\mathcal{S}^{(1)}_{-1} = \frac{1}{2}\left(v'_1 + \sqrt{3}\mathcal{S}'_2\right) \quad (\mathrm{S}^3 = +1/2) \tag{3.25a}$$

$$\mathcal{S}^{(2)}_{-1} = \frac{1}{2}\left(v'_5 + \sqrt{3}\mathcal{S}''_2\right) \quad (\mathrm{S}^3 = +3/2) \tag{3.25b}$$

are those corresponding to $m_5 = -1$. Then, to sum up :

- $\mathrm{D}^{(1)}_{1/3}$, $\mathrm{D}^{(2)}_{1/3}$, $\mathcal{S}^{(1)}_{1/3}$ and $\mathcal{S}^{(1)}_{1/3}$ generate the basis of $E_{1/3}$, and
- $\mathcal{S}^{(1)}_{-1}$ and $\mathcal{S}^{(2)}_{-1}$ generate that of E_{-1}.

The expressions of the various eigenspinors thus defined are given explicitly in terms of Dirac spinors in Appendix A. In Appendix B, some of them are reexpressed by means of tensor-spinors of the Rarita-Schwinger type and compared to analogous expressions given in Ref. [**8**].

The remarkable fact already mentioned in the Introduction is the role played by negative-energy Dirac spinors. They appear as an unavoidable ingredient in the construction of the irreducible representations of the Lorentz spin group. As a result, the doublet $\mathcal{D}_1$ associated to the standard spinor form (1.1) does have a projection in $D(\frac{1}{2},1) \oplus D(1,\frac{1}{2})$ (more precisely in $E_{-5/3}$) since from (3.20) :

$$\mathcal{D}_1 = \frac{1}{2}\left\{\mathrm{D}_1^{(1)} + \sqrt{\frac{3}{2}}\left(\mathrm{D}_1^{(2)} + \mathrm{D}_{-5/3}^{(1)}\right)\right\} \tag{3.26}$$

As discussed in the Introduction, this makes its transformation under a Lorentz boost quite different from that of a *genuine* spin 1/2 spinor wave function belonging to a representation $D(\frac{1}{2},0) \oplus D(0,\frac{1}{2})$, like a Dirac spinor. We now know that such a *genuine* spin 1/2 three-quark wave function should be constructed as a linear combination involving only the components of the two doublets $\mathrm{D}_1^{(1)}$, $\mathrm{D}_1^{(2)}$ and the antisymmetric singlet $\mathcal{A}$. Let us add that the emergence of an antisymmetric singlet is possible just because negative-energy Dirac spinors are taken into account. Finally, let us mention that the spinor wave functions here derived for a three-quark S state may serve as a basis to construct wave functions of orbitally excited three-quark states by suitable combinations with moments of the relative momenta.

4. THE SPIN-FLAVOR WAVE FUNCTION OF THE PROTON

In the three-quark picture of the proton the flavor content of that particle is described by the well-known wave functions :

$$\xi_{ijk} = \tfrac{1}{\sqrt{2}}(\delta_{iu}\delta_{jd} - \delta_{id}\delta_{ju})\delta_{ku} \tag{4.1a}$$

$$\eta_{ijk} = \tfrac{1}{\sqrt{6}}\left\{(\delta_{iu}\delta_{jd} + \delta_{id}\delta_{ju})\delta_{ku} - 2\delta_{iu}\delta_{ju}\delta_{kd}\right\} \tag{4.1b}$$

which, taken together, form a doublet under the permutation group S_3 (here, the labels u and d refer to u and d quark-flavors).

Wave functions are expanded in terms of products of spin, flavor and momentum distributions. Then, to obtain a completely symmetric proton wave function, one couples the two above flavor wave functions with the two components a and s of a third-rank spinor doublet (of spin $J = 1/2$) into the inner product $a\xi + s\eta$ which is S_3 invariant. More specifically, one sets :

$$\varphi(I,J,K) \sim (a\xi + s\eta)F(P.p_\xi, P.p_\eta, p_\xi^2, p_\eta^2, p_\eta.p_\xi) \tag{4.2}$$

where the momentum-distribution amplitude F must be S_3 invariant too. Actually, in standard covariant approaches, a and s are taken respectively as a_1 (Eq.3.13) and s_1 (Eq.3.17).

However, more general constructions may be considered, if it happens that spin-flavor and momentum distributions can mix together differently. In that case the momentum distribution need not be completely symmetric any longer. Then, noticing that the four-momenta p_ξ and $-\frac{\sqrt{3}}{2}p_\eta$ form jointly a doublet under S_3, we see that we may form, for instance, polynomials in (Lorentz-invariant) momentum variables, which transform under S_3 like doublets, symmetric singlets or antisymmetric singlets and which, when coupled to spin-flavor wave functions showing the appropriate S_3 symmetry, also lead to a completely symmetric overall wave function. Let us remark that, in the relativistic harmonic oscillator model [**7**], such polynomials appear in the momentum distributions of some excited three-quark states, which have been associated with nucleon resonances (of spin 1/2). On the other hand, the proton wave function derived from QCD sum rules by Chernyak and Zhitnitsky [**11**] is asymmetric in momentum distribution. This is an encouragement for us to search for wave functions of forms more general than (4.2).

Consider first the variables p_ξ^2 , $-\frac{\sqrt{3}}{2}p_\eta.p_\xi$ and $\frac{3}{4}p_\eta^2$. They may be arranged into the S_3 doublet :

$$\left(\tilde{a} = -\frac{\sqrt{3}}{2}p_\eta.p_\xi \quad , \quad \tilde{s} = p_\xi^2 - \frac{3}{4}p_\eta^2\right) \tag{4.3a}$$

and the S_3-symmetric singlet :

$$z = p_\xi^2 + \frac{3}{4}p_\eta^2 \tag{4.3b}$$

Let us assume for a while that condition (2.6) is to be replaced by the more restrictive ones :

$$\frac{1}{2}\varepsilon_{\mu\nu\rho\sigma}P^\nu \mathrm{L}^{(\xi)\rho\sigma}\varphi(I,J,K) = \frac{1}{2}\varepsilon_{\mu\nu\rho\sigma}P^\nu \mathrm{L}^{(\eta)\rho\sigma}\varphi(I,J,K) = 0 \tag{4.4}$$

meaning that the proton should be an S state as regards *all* relative angular momenta. In that case, the invariant $p_\xi.p_\eta$ should not appear at all in the proton wave function, as also the invariant $p_\xi^2 - \frac{3}{4}p_\eta^2$ which belongs to the same doublet. Thus, under assumption (4.4), the momentum distribution depends on the momenta variables (4.3) only through the symmetric-singlet combination z.

Next, regarding the dependence on $z_1 = P.p_\xi$ and $z_2 = -\frac{\sqrt{3}}{2}P.p_\eta$, the following general features may be brought out. Let us assume we can expand the momentum-distribution amplitudes in series of polynomials in z_1 and z_2. To classify those polynomials according to S_3, it appears very convenient to consider them in the context of the well-known formalism where all irreducible representations of the group $SU(2)$ are defined as polynomials in the two (complex) components c_1 and c_2 of the 2-spinor $\left(\begin{array}{c} c_1 \\ c_2 \end{array}\right)$ (see, for instance, Ref. [**15**]).

In that framework, the irreducible representation space corresponding to the spin j is spanned by the $2j+1$ monomials

$$v^j_m = \frac{c_1^{j+m} c_2^{j-m}}{\sqrt{(j+m)!(j-m)!}} \tag{4.5}$$

of weight (i.e. total degree) $N = 2j$ $(m = -j, \ -j+1, \ ..., \ j)$.

Indeed, we may view the doublet $\widetilde{\mathcal{D}} = \left(\begin{array}{c} z_1 \\ z_2 \end{array}\right)$ as a 2-spinor which, under the permutations of S_3, undergoes linear transformations as given by :

$$\mathcal{P}_{12}\widetilde{\mathcal{D}} = \left(\begin{array}{cc} -1 & 0 \\ 0 & 1 \end{array}\right)\widetilde{\mathcal{D}} = -\sigma_3\widetilde{\mathcal{D}} \tag{4.6a}$$

$$\mathcal{P}_{13}\widetilde{\mathcal{D}} = \tfrac{1}{2}\left(\begin{array}{cc} 1 & \sqrt{3} \\ \sqrt{3} & -1 \end{array}\right)\widetilde{\mathcal{D}} = \tfrac{1}{2}\left(\sigma_3 + \sqrt{3}\sigma_1\right)\widetilde{\mathcal{D}} \tag{4.6b}$$

$$\mathcal{P}_{12}\mathcal{P}_{13}\widetilde{\mathcal{D}} = \tfrac{1}{2}\left(\begin{array}{cc} -1 & \sqrt{3} \\ -\sqrt{3} & -1 \end{array}\right)\widetilde{\mathcal{D}} = \tfrac{1}{2}\left(-1 + i\sqrt{3}\sigma_2\right)\widetilde{\mathcal{D}} \tag{4.6c}$$

where 2×2 Pauli matrices σ_r $(r = 1, \ 2, \ 3)$ have been introduced. Since in 2-spinor space a rotation $\mathcal{R}_{\vec{n}}(\theta)$ of angle θ about the direction $\vec{n}$ is represented by the matrix $\mathrm{R}_{\vec{n}}(\theta) = \cos(\theta/2) + i\ \sin(\theta/2)(\vec{n}.\vec{\sigma})$, formulas (4.6) lead immediately to the interpretation of permutations in terms of rotations in three-dimensional space :

- $\mathcal{P}_{12}$ represents a rotation of angle π about the z-axis, times i,
- $\mathcal{P}_{13}$ a rotation of angle π about the axis $(\sqrt{3}/2, \ 0 \ , \ 1/2)$, times $-i$, and
- $\mathcal{P}_{12}\mathcal{P}_{13}$ a rotation of angle $4\pi/3$ about the y-axis.

To derive the dimensions of irreducible representations of S_3 in the space of polynomials of weight N, we only have to compute, here again, the traces of representatives of Young idempotents (3.2) in that space[7]. The trace of the representative R^j of a rotation $\mathcal{R}_{\vec{n}}(\theta)$ is easily performed using a basis of eigenvectors of the spin-projection on $\vec{n}$. This yields :

$$\mathrm{Tr}\ \mathrm{R}^j = \sum_{-j}^{+j} e^{im\theta} = \frac{\sin\{(N+1)\theta/2\}}{\sin(\theta/2)} \tag{4.7}$$

[7]The so-called *characters* in group theory (see, for instance, Ref. [**16**]).

Hence :

$$\text{Tr } \mathcal{P}_{12} = \text{Tr } \mathcal{P}_{13} = \text{Tr } \mathcal{P}_{23} = \text{R}_N = \tfrac{1}{2}\left(1 + (-1)^N\right) \tag{4.8a}$$

$$\text{Tr } (\mathcal{P}_{12}\, \mathcal{P}_{13}) = \text{Tr } (\mathcal{P}_{13}\, \mathcal{P}_{12}) = \text{R}'_N = \frac{\sin\left\{(N+1)\frac{2\pi}{3}\right\}}{\sin(\frac{2\pi}{3})} \tag{4.8b}$$

and :

$$\text{Tr } \mathcal{Y}_D = 2\text{Tr } \mathcal{Y}_{M(A)} = 2\text{Tr } \mathcal{Y}_{M(S)} = d_D = \tfrac{2}{3}\,(N + 1 - \text{R}'_N) \tag{4.9a}$$

$$\text{Tr } \mathcal{Y}_S = d_S = \tfrac{1}{6}\,(N + 1 + 3\text{R}_N + 2\text{R}'_N) \tag{4.9b}$$

$$\text{Tr } \mathcal{Y}_A = d_A = \tfrac{1}{6}\,(N + 1 - 3\text{R}_N + 2\text{R}'_N) \tag{4.9c}$$

Notice that $d_S = d_A$ if N is odd, while $d_S = d_A + 1$ if N is even. The values of the various d up to $N = 15$ are listed in Table III. This Table seems to indicate that more and more doublets and singlets of the same weight N appear as N is increasing. Fortunately, in spite of this apparent complexity, there is a simple feature that is revealed by a detailed inspection of formulas (4.9) ; namely, d_S is identical to the number r_N of partitions of N, when N is expressed as :

$$N = 2p + 3q \tag{4.10a}$$

p and q being non-negative integers. As can be easily checked, r_N may be computed from :

$$r_N = \frac{1}{N!}\left[\frac{d^N}{dx^N}\left(\frac{1}{(1-x^2)(1-x^3)}\right)\right]_{x=0} \tag{4.10b}$$

The direct consequence of (4.10a) is that any symmetric singlet defined as a polynomial in z_1 and z_2 is in fact a polynomial in only the following two independent fundamental symmetric singlets (see Appendix C) :

$$w_1 = z_1^2 + z_2^2 = (P.p_\xi)^2 + \tfrac{3}{4}(P.p_\eta)^2 \tag{4.11a}$$

$$w_2 = z_2(z_2^2 - 3z_1^2) \tag{4.11b}$$

$$\equiv -\tfrac{\sqrt{3}}{18}\,(M^2 - 3(P.p_1)^2)\,(M^2 - 3(P.p_2)^2)\,(M^2 - 3(P.p_3)^2)$$

which have (mass)^2 dimension two and three respectively. Actually, that result may be generalized to any symmetric-singlet function $f(z_1, z_2)$. A possible proof is presented in Appendix C.

Next, we find :

$$d_A = r_{N-3} \quad (N \geq 3) \tag{4.12a}$$

$$d_D = 2(r_{N-1} + r_{N-2}) \quad (N \geq 3) \tag{4.12b}$$

i.e., p and q being two non-negative integers, d_A is the number of ways N may be expressed as :

$$N = 3 + 2p + 3q \tag{4.13}$$

while the number of doublets $d_D/2$ of weight N is also the total number of ways N may be expressed as either :

$$N = 1 + 2p + 3q \tag{4.14a}$$

or :

$$N = 2 + 2p + 3q \tag{4.14b}$$

From the preceding analysis, it follows that any polynomial antisymmetric singlet has the form of the product of the fundamental antisymmetric singlet of lowest weight $N = 3$ (see Appendix C) :

$$\tilde{\mathcal{A}} = z_1(z_1^2 - 3z_2^2) \tag{4.15}$$

by a polynomial in w_1 and w_2 of weight $N-3$ in the z_i. Similarly, any polynomial doublet of weight N is found to be a linear combination of two doublets, one being the product of the first fundamental doublet which has weight $N = 1$:

$$\widetilde{\mathcal{D}}_1 = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \tag{4.16a}$$

by a polynomial in w_1 and w_2 of weight $N-1$ in the z_i, the other being the product of the second fundamental doublet which has weight $N = 2$:

$$\widetilde{\mathcal{D}}_2 = \begin{pmatrix} 2z_1z_2 \\ z_1^2 - z_2^2 \end{pmatrix} \tag{4.16b}$$

by a polynomial in w_1 and w_2 of weight $N-2$ in the z_i.

The irreducible representations of S_3 by polynomials in z_1 and z_2 are given explicitly in Appendix C, up to $N = 6$.

Bound-state wave functions are usually assumed to display regular behaviour. We may then safely generalize all the above results to such functions and state that, regarding its dependence on z_1 and z_2, and whatever its explicit shape, the momentum-distribution part of the proton wave function should be constructed from the following S_3-irreducible representations :

- symmetric singlets $\tilde{\mathcal{S}}$, which are functions of w_1 and w_2 (4.11) only,
- antisymmetric singlets which are expressed as products of $\tilde{\mathcal{A}}$ (4.15) by (symmetric singlets) functions of w_1 and w_2,

- doublets $\tilde{\mathcal{D}}$ which have the form $\tilde{\mathcal{D}}_1 F_1(w_1, w_2) + \tilde{\mathcal{D}}_2 F_2(w_1, w_2)$, where $\tilde{\mathcal{D}}_1$ and $\tilde{\mathcal{D}}_2$ are given by (4.16), F_1 and F_2 being two symmetric-singlet functions.

Symmetric singlets $\tilde{\mathcal{S}}$ alone lead to wave functions of the familiar form (4.2). However, if doublets $\tilde{\mathcal{D}}$ and the antisymmetric singlet $\tilde{\mathcal{A}}$ must be taken into account, the spin-flavor distribution may prove more intricate than that of (4.2). In particular, it may then involve the symmetric- and antisymmetric- singlet spinors defined in the preceding section. This becomes possible in that case just because the coupling of the flavor doublet $\mathcal{D}_{\mathrm{f}} = \begin{pmatrix} \xi \\ \eta \end{pmatrix}$ with a doublet $\tilde{\mathcal{D}} = \begin{pmatrix} \tilde{a} \\ \tilde{s} \end{pmatrix}$ yields not only the doublet :

$$\mathcal{D}_{\mathrm{fm}} = \begin{pmatrix} a_{\mathrm{fm}} = \xi\tilde{s} + \eta\tilde{a} \\ s_{\mathrm{fm}} = \xi\tilde{a} - \eta\tilde{s} \end{pmatrix} \tag{4.17a}$$

but also the symmetric singlet :

$$\mathcal{S}_{\mathrm{fm}} = \xi\tilde{a} + \eta\tilde{s} \tag{4.17b}$$

and the antisymmetric singlet :

$$\mathcal{A}_{\mathrm{fm}} = \xi\tilde{s} - \eta\tilde{a} \tag{4.17c}$$

which, in their turn, should be coupled in a S_3-invariant way to third-rank spinors of the same symmetry (here, the label “fm” refers to “flavor-momentum” coupling). Let us list the various possible constructions we now get.

1) A doublet of third-rank spinors $\mathcal{D} = \begin{pmatrix} a \\ s \end{pmatrix}$ may be coupled to either :

i) a doublet $\mathcal{S}\mathcal{D}_{\mathrm{f}}$ in the standard form (4.2),

ii) or a doublet $\mathcal{D}_{\mathrm{fm}}$ in the form :

$$\varphi_1'(I, J, K) = a a_{\mathrm{fm}} + s s_{\mathrm{fm}} \equiv \xi(a\tilde{s} + s\tilde{a}) + \eta(a\tilde{a} - s\tilde{s}) \tag{4.18a}$$

2) A symmetric singlet, third-rank spinor $\mathcal{S}$ should be coupled to a symmetric singlet $\mathcal{S}_{\mathrm{fm}}$ in the form :

$$\varphi_2'(I, J, K) = \mathcal{S}\mathcal{S}_{\mathrm{fm}} = \mathcal{S}(\xi\tilde{a} + \eta\tilde{s}) \tag{4.18b}$$

3) The antisymmetric singlet, third-rank spinor $\mathcal{A}$ is to be coupled to an antisymmetric singlet $\mathcal{A}_{\mathrm{fm}}$ in the form :

$$\varphi_3'(I, J, K) = \mathcal{A}\mathcal{A}_{\mathrm{fm}} = \mathcal{A}(\xi\tilde{s} - \eta\tilde{a}) \tag{4.18c}$$

Under condition (4.4), the most general wave function of a spin 1/2 three-quark composite particle is thus a linear combination of the forms (4.2) and (4.17). Notice that the symmetric-singlet momentum-distribution amplitudes involved in that combination may also depend on the momentum variable z (4.3b).

Things become even more complicated if condition (4.4) is replaced by the less restrictive one (2.6) since then both variables $z'_1 = -\frac{\sqrt{3}}{2}p_\eta.p_\xi$ and $z'_2 = p_\xi^2 - \frac{3}{4}p_\eta^2$ may appear in the wave function. If we were dealing with those variables only, we would get, proceeding as above, the following S_3-irreducible fundamental representations :

- two symmetric singlets :

$$w'_1 = z'^2_1 + z'^2_2 \quad , \quad w'_2 = z'_2(z'^2_2 - 3z'^2_1) \tag{4.19a}$$

- two doublets :

$$\tilde{\mathcal{D}}'_1 = \begin{pmatrix} z'_1 \\ z'_2 \end{pmatrix} \quad , \quad \tilde{\mathcal{D}}'_2 = \begin{pmatrix} 2z'_1z'_2 \\ z'^2_1 - z'^2_2 \end{pmatrix} \tag{4.19b}$$

- and the antisymmetric singlet :

$$\tilde{\mathcal{A}}' = z'_1(z'^2_1 - 3z'^2_2) \tag{4.19c}$$

Coupling these new objects, in a non-singular way, with the analogous ones defined through the variables z_1 and z_2, will provide us with the most general S_3-irreducible regular forms that a Lorentz-invariant momentum distribution of a three-quark system may take on. All possible primary forms we now get are :

- Ten symmetric-singlets, namely, w_1, w_2, w'_1, w'_2, z, and five other ones obtained by coupling doublets or antisymmetric singlets :

$$w_{11} = \tilde{\mathcal{D}}_1.\tilde{\mathcal{D}}'_1 = z_1z'_1 + z_2z'_2 \tag{4.20a}$$

$$w_{12} = \tilde{\mathcal{D}}_1.\tilde{\mathcal{D}}'_2 = 2z_1z'_1z'_2 + z_2(z'^2_1 - z'^2_2) \tag{4.20b}$$

$$w_{21} = \tilde{\mathcal{D}}_2.\tilde{\mathcal{D}}'_1 = 2z'_1z_1z_2 + z'_2(z^2_1 - z^2_2) \tag{4.20c}$$

$$\begin{aligned} w_{22} = \tilde{\mathcal{D}}_2.\tilde{\mathcal{D}}'_2 &= 4z_1z_2z'_1z'_2 + (z^2_1 - z^2_2)(z'^2_1 - z'^2_2) \\ &\equiv (z_1z'_1 + z_2z'_2)^2 - (z_1z'_2 - z_2z'_1)^2 \end{aligned} \tag{4.20d}$$

$$w_a = \tilde{\mathcal{A}}\tilde{\mathcal{A}}' = z_1z'_1(z^2_1 - 3z^2_2)(z'^2_1 - 3z'^2_2) \tag{4.20e}$$

- Twelve doublets : the four ones $\tilde{\mathcal{D}}_1$, $\tilde{\mathcal{D}}_2$, $\tilde{\mathcal{D}}'_1$ and $\tilde{\mathcal{D}}'_2$ are already known; four other ones are obtained by coupling $\tilde{\mathcal{D}}_1$ and $\tilde{\mathcal{D}}_2$ successively with $\tilde{\mathcal{D}}'_1$ and $\tilde{\mathcal{D}}'_2$ (in the same way as in formula (4.16a)) :

$$\tilde{\mathcal{D}}_{11} = \begin{pmatrix} z_1z'_2 + z'_1z_2 \\ z_1z'_1 - z_2z'_2 \end{pmatrix} \quad , \quad \tilde{\mathcal{D}}_{12} = \begin{pmatrix} z_1(z'^2_1 - z'^2_2) + 2z_2z'_1z'_2 \\ 2z_1z'_1z'_2 - z_2(z'^2_1 - z'^2_2) \end{pmatrix} \tag{4.21a}$$

$$\tilde{\mathcal{D}}_{21} = \begin{pmatrix} 2z'_2z_1z_2 + z'_1(z^2_1 - z^2_2) \\ 2z_1z_2z'_1 - z'_2(z^2_1 - z^2_2) \end{pmatrix} \tag{4.21b}$$

$$\tilde{\mathcal{D}}_{22} = \begin{pmatrix} 2z_1z_2(z'^2_1 - z'^2_2) + 2z'_1z'_2(z^2_1 - z^2_2) \\ 4z_1z_2z'_1z'_2 - (z^2_1 - z^2_2)(z'^2_1 - z'^2_2) \end{pmatrix} \tag{4.21c}$$

The last four doublets are constructed as follows :

$$\tilde{\mathcal{D}}_{a1} = \tilde{\mathcal{A}}' \begin{pmatrix} z_2 \\ -z_1 \end{pmatrix} \quad , \quad \tilde{\mathcal{D}}_{a2} = \tilde{\mathcal{A}}' \begin{pmatrix} z_1^2 - z_2^2 \\ -2z_1 z_2 \end{pmatrix} \tag{4.21d}$$

$$\tilde{\mathcal{D}'}_{a1} = \tilde{\mathcal{A}} \begin{pmatrix} z'_2 \\ -z'_1 \end{pmatrix} \quad , \quad \tilde{\mathcal{D}'}_{a2} = \tilde{\mathcal{A}} \begin{pmatrix} z_1'^2 - z_2'^2 \\ -2z'_1 z'_2 \end{pmatrix} \tag{4.21e}$$

- Six antisymmetric singlets : $\tilde{\mathcal{A}}$, $\tilde{\mathcal{A}}'$ and :

$$\tilde{\mathcal{A}}_{11} = (\tilde{\mathcal{D}}_1 \wedge \tilde{\mathcal{D}'}_1) = z_1 z'_2 - z'_1 z_2 \tag{4.22a}$$

$$\tilde{\mathcal{A}}_{12} = (\tilde{\mathcal{D}}_1 \wedge \tilde{\mathcal{D}'}_2) = z_1(z_1'^2 - z_2'^2) - 2z_2 z'_1 z'_2 \tag{4.22b}$$

$$\tilde{\mathcal{A}}_{21} = (\tilde{\mathcal{D}'}_1 \wedge \tilde{\mathcal{D}}_2) = z'_1(z_1^2 - z_2^2) - 2z'_2 z_1 z_2 \tag{4.22c}$$

$$\tilde{\mathcal{A}}_{22} = (\tilde{\mathcal{D}}_2 \wedge \tilde{\mathcal{D}'}_2) = 2z_1 z_2(z_1'^2 - z_2'^2) - 2z'_1 z'_2(z_1^2 - z_2^2) \tag{4.22d}$$

However, the new forms obtained in this way are not all independent : most of them turn out to have a functional dependence on others through either a polynomial form or a rational form. Of course, the criterion of independence to be applied depends on the class of functions to which the momentum distribution actually belongs. Let us first assume the latter to be a polynomial in the four variables z_1, z_2, z'_1 and z'_2. Then we have the following general feature.

Consider, as above, the space of all polynomials of weight $n = 2j$ in z_1 and z_2 as an irreducible representation $D(j)$ of the group $SU(2)$, which is spanned by the $2j + 1$ monomials :

$$u^j_m = \frac{z_1^{j+m} z_2^{j-m}}{\sqrt{(j+m)!(j-m)!}} \tag{4.23a}$$

with $m = -j, \ -j+1, ..., j$.

Similarly, the space of all polynomials of weight $n' = 2j'$ in z'_1 and z'_2 may be viewed as an irreducible representation $D(j')$ of $SU(2)$, which is spanned by the $2j' + 1$ monomials :

$$u'^{j'}_{m'} = \frac{{z'}_1^{j'+m'} {z'}_2^{j'-m'}}{\sqrt{(j'+m')!(j'-m')!}} \tag{4.23b}$$

with $m' = -j', \ -j'+1, ..., j'$.

Coupling the first polynomials with the second ones then amounts to perform the Kronecker product $K = D(j) \times D(j')$. That space K reduces into the direct sum of irreducible representations $D(J)$ of $SU(2)$, with $J = |j-j'|, \ |j-j'|+1, ..., j+j'$. Accordingly, we have the Clebsch-Gordan decomposition :

$$u^j_m \ u'^{j'}_{m'} = \sum_J \sum_{M=-J}^{M=J} < j \ m \ j' \ m' \ |J \ M > \Psi^J_M \tag{4.23c}$$

where $< j\ m\ j'\ m'\ |J\ M >$ are Clebsch-Gordan coefficients and Ψ^J_M the basis functions of $D(J)$. It can be shown (see, for instance, Morton Hamermesh, Ref. [**14**], § 9-8) that each Ψ^J_M may be expressed as the product :

$$\Psi^J_M = \varphi^J_M\ F_J \tag{4.23d}$$

of a polynomial φ^J_M of weight $2J$ in $z_1,\ z_2,\ z'_1,\ z'_2$, which transforms according to $D(J)$, by the $SU(2)$-invariant function :

$$F_J = (z_1\ z'_2 - z'_1\ z_2)^{j+j'-J} \tag{4.23e}$$

which is a polynomial in $z_1,\ z_2,\ z'_1$ and z'_2 of weight $2(j+j'-J) = 2s$, s being a non-negative integer. Notice that F_J is also a singlet under S_3 (symmetric if s is even, antisymmetric if s is odd). Now, the traces of the representative in $D(J)$ of the Young idempotents of S_3 are given by formulas (4.8), where N should be simply replaced by $\nu = 2J$. From formulas (4.9-13) and corresponding discussions, and taking Eqs.(4.23) into account, we may state the following. For a given representation $D(J)$, the number of independent S_3 singlets, either symmetric or antisymmetric, is also the number of partitions of ν when that integer is expressed as $\nu = 2p + 3q$, p and q being two non-negative integers. This also means that in $D(J)$, S_3-singlet polynomials of weight $n + n' = 2s + \nu$ in the four variables $z_1,\ z_2,\ z'_1$ and z'_2 are expressible as polynomials in polynomial singlets of weights 2 or 3 in these variables. On the other hand, the number of independent doublets in $D(J)$ is the number of ways ν may be expressed as either $\nu = 1 + 2p + 3q$ or $\nu = 2 + 2p + 3q$. Hence, we may conclude that S_3 doublets are linear combinations (with symmetric-singlet polynomials as coefficients) of polynomial doublets of weights 1 or 2. Since these results hold for every independent $D(J)$, they also apply to the whole space K of polynomials of weight $N = n + n'$ in the $z_i,\ {z'}_i$. Indeed, it may be easily checked that all the polynomial forms of weights 4, 5 or 6 (or even 3 in the case of doublets) in Eq.(4.20-22) can be reexpressed as polynomials in S_3-irreducible forms of weights not greater than 3 (not greater than 2 in the case of doublets). The better way to see this is to set, as in Appendix C :

$$z_2 = \sqrt{w_1}\cos\theta \quad , \quad z_1 = \sqrt{w_1}\sin\theta \tag{4.24a}$$

$$z'_2 = \sqrt{w'_1}\cos\theta' \quad , \quad z'_1 = \sqrt{w'_1}\sin\theta' \tag{4.24b}$$

For instance, in the symmetric-singlet sector we have :

$$w_2 = (w_1)^{3/2}\cos(3\theta) \quad , \quad w'_2 = (w'_1)^{3/2}\cos(3\theta') \tag{4.25a}$$

$$w_{11} = \sqrt{w_1 w'_1}\cos(\theta - \theta') \quad , \quad w_{12} = -w'_1\sqrt{w_1}\cos(2\theta' + \theta) \tag{4.25b}$$

$$w_{21} = -w_1\sqrt{w'_1}\cos(2\theta + \theta') \quad , \quad w_{22} = w_1 w'_1 \cos\left[2(\theta - \theta')\right] \tag{4.25c}$$

$$w_a = (w_1 w'_1)^{3/2}\sin(3\theta)\sin(3\theta') \tag{4.25d}$$

Among these variables, only w_{22} and w_a are expressible as polynomials in the others :

$$w_{22} = w_1 w_1' \,[2\cos^2(\theta-\theta') - 1] = 2w_{11}^2 - w_1 w_1' \tag{4.26a}$$

$$\begin{aligned} w_a = (w_1 w_1')^{3/2}\,[4\cos^3(\theta-\theta') - 3\cos(\theta-\theta') + \\ -\cos(3\theta)\cos(3\theta')] = 4w_{11}^3 - 3w_1 w_1' w_{11} - w_2 w_2' \end{aligned} \tag{4.26b}$$

It follows that symmetric-singlet polynomials may in fact be rewritten, in a *regular* way, as polynomials in the eight following forms of weights 2 or 3 : w_1, w_1', w_2, w_2', w_{11}, w_{12}, w_{21} and z.

Regarding doublets, we have the relations :

$$\begin{aligned} \tilde{\mathcal{D}}_{22} = -w_1 w_1' \begin{pmatrix} \sin[2(\theta+\theta')] \\ \cos[2(\theta+\theta')] \end{pmatrix} = 2w_{12}\tilde{\mathcal{D}}_1 - w_1 \tilde{\mathcal{D}'}_2 \\ = 2w_{21}\tilde{\mathcal{D}'}_1 - w_1' \tilde{\mathcal{D}}_2 \end{aligned} \tag{4.27a}$$

$$\tilde{\mathcal{D}}_{a1} = -w_1 (w_1')^{3/2} \sin(3\theta') \begin{pmatrix} \cos\theta \\ -\sin\theta \end{pmatrix} = w_1' \tilde{\mathcal{D}}_{11} - w_2' \tilde{\mathcal{D}}_1 - 2w_{11}\tilde{\mathcal{D}'}_2 \tag{4.27b}$$

$$\tilde{\mathcal{D}}_{12} = -w_1' \tilde{\mathcal{D}}_1 + 2w_{11}\tilde{\mathcal{D}'}_1 \tag{4.27c}$$

$$\tilde{\mathcal{D}}_{21} = -w_1 \tilde{\mathcal{D}'}_1 + 2w_{11}\tilde{\mathcal{D}}_1 \tag{4.27d}$$

$$\tilde{\mathcal{D}}_{a2} = w_1 w_1' \tilde{\mathcal{D}'}_1 - w_2' \tilde{\mathcal{D}}_2 - 2w_{12}\tilde{\mathcal{D}}_{11} \tag{4.27e}$$

$$\tilde{\mathcal{D}'}_{a1} = w_1 \tilde{\mathcal{D}}_{11} - w_2 \tilde{\mathcal{D}'}_1 - 2w_{11}\tilde{\mathcal{D}}_2 \tag{4.27f}$$

$$\tilde{\mathcal{D}'}_{a2} = w_1 w_1' \tilde{\mathcal{D}}_1 - w_2 \tilde{\mathcal{D}'}_2 - 2w_{21}\tilde{\mathcal{D}}_{11} \tag{4.27g}$$

In that case, we get the result that any S_3-doublet polynomial is a *regular* linear combination of the five doublets $\tilde{\mathcal{D}}_1$, $\tilde{\mathcal{D}}_2$, $\tilde{\mathcal{D}'}_1$, $\tilde{\mathcal{D}'}_2$ and $\tilde{\mathcal{D}}_{11}$, of weights 1 or 2.

Finally, for antisymmetric-singlet polynomials, we find the relation :

$$\tilde{\mathcal{A}}_{22} = -2w_{11}\tilde{\mathcal{A}}_{11} \tag{4.28}$$

and that S_3-antisymmetric singlet polynomials are *regular* linear combinations of the five following ones : $\tilde{\mathcal{A}}$, $\tilde{\mathcal{A}}'$, $\tilde{\mathcal{A}}_{11}$, $\tilde{\mathcal{A}}_{12}$ and $\tilde{\mathcal{A}}_{21}$ which are of weights 2 or 3.

As already mentioned, the coefficients of the linear combinations here considered are themselves symmetric-singlet polynomials.

Thus, for the restricted class of polynomials in the z_i, z_i', S_3-irreducible polynomials of weights not greater than 3 constitute the basic forms in terms of which every polynomial is expressible in the same algebraic fashion, i.e. is a polynomial in these basic variables.

But this is not the end of the story. Actually, some of the above "basic" polynomials are expressible, in their turn, as regular rational functions of other "basic" polynomials, which appear then as more fundamental than the former. For instance, using the trivial trigonometric relations :

$$\cos(2\theta' + \theta) = \frac{\cos(3\theta) + 2\cos(3\theta')\cos(\theta - \theta')}{4\cos^2(\theta - \theta') - 1} \tag{4.29a}$$

$$\cos(2\theta + \theta') = \frac{\cos(3\theta') + 2\cos(3\theta)\cos(\theta - \theta')}{4\cos^2(\theta - \theta') - 1} \tag{4.29b}$$

we readily get :

$$w_{12} = -\frac{w_2 w_1'^2 + 2w_1 w_2' w_{11}}{4w_{11}^2 - w_1 w_1'} \tag{4.30a}$$

$$w_{21} = -\frac{w_2' w_1^2 + 2w_1' w_2 w_{11}}{4w_{11}^2 - w_1 w_1'} \tag{4.30b}$$

In an analogous way, using :

$$\sin(2\theta' + \theta) = \frac{\sin(3\theta) + 2\sin(3\theta')\cos(\theta - \theta')}{4\cos^2(\theta - \theta') - 1} \tag{4.31a}$$

$$\sin(2\theta + \theta') = \frac{\sin(3\theta') + 2\sin(3\theta)\cos(\theta - \theta')}{4\cos^2(\theta - \theta') - 1} \tag{4.31b}$$

$$\sin(\theta - \theta') = \frac{\sin\left[3(\theta - \theta')\right]}{4\cos^2(\theta - \theta') - 1} \tag{4.31c}$$

we obtain :

$$\mathcal{A}_{12} = \frac{w_1'^2 \mathcal{A} + 2w_1 w_{11} \mathcal{A}'}{4w_{11}^2 - w_1 w_1'} \tag{4.32a}$$

$$\mathcal{A}_{21} = \frac{w_1^2 \mathcal{A}' + 2w_1' w_{11} \mathcal{A}}{4w_{11}^2 - w_1 w_1'} \tag{4.32b}$$

$$\mathcal{A}_{11} = \frac{-w_2' \mathcal{A} + w_2 \mathcal{A}'}{4w_{11}^2 - w_1 w_1'} \tag{4.32c}$$

As for the doublets, we have the relation :

$$\tilde{\mathcal{D}}_{11} = \frac{1}{2w_{11}} \left(\tilde{\mathcal{D}}_2 + \tilde{\mathcal{D}'}_2 \right) \tag{4.33}$$

Thus, extending our analysis from the class of polynomials to a wider class of regular functions so as to include with it the preceding rational expressions, we are now left with only six basic symmetric singlets : w_1, w_2, w_1', w_2', w_{11} and z, two basic antisymmetric singlets : $\mathcal{A}$ and $\mathcal{A}'$, and four basic doublets : $\tilde{\mathcal{D}}_1$, $\tilde{\mathcal{D}}_2$, $\tilde{\mathcal{D}'}_1$ and $\tilde{\mathcal{D}'}_2$.

In Appendix C we also present a possible proof of the following general proposition : any S_3-symmetric singlet $f(z_1, z_2, z'_1, z'_2)$ is in fact a functional in only the five basic symmetric singlets w_1, w_2, w'_1, w'_2 and w_{11}.

Next, the S_3-irreducible momentum-distribution amplitudes constructed from the above-defined basic forms should be coupled with appropriate spin-flavor amplitudes according to the same scheme as in formulas (4.2) and (4.17).

The formalism we have developed so far represents the most general, model-independent statements we can make, within the constituent-quark picture, about covariant extensions of wave functions of three-quark S states of spin 1/2 (or spin 3/2 as well). Indeed, the general structures we have found are rather complicated ; but in the real case, wave functions may prove simpler than that. As usual, experimental data should be our guide, here in order to eventually discriminate between all possible forms those wave functions may take on, taking into account general features such as, for instance, power-falloff laws of form factors. However, a comparison with the data cannot be completely model-independent. Indeed, this requires explicit calculations of transition amplitudes involving wave functions of composite hadrons, which, in the absence of a well established theory of confinement dynamics and of a general description of interactions between compound systems[8], can only be performed using specific effective models (like that of Refs. [**6, 7**]). We will not elaborate further on that question which is beyond the scope of the present work.

5. GAMMA MATRICES IN E AND RELATIVISTIC WAVE EQUATION FOR THE PROTON

One usually calls "gamma matrices" or "Dirac matrices" any set of four matrices λ_μ transforming like a 4-vector under the Lorentz group and verifying the anticommutation relations :

$$\lambda_\mu \lambda_\nu + \lambda_\nu \lambda_\mu = 2 g_{\mu\nu} \tag{5.1}$$

We need hardly recall the essential role played by such matrices in the theory of relativistic wave equations : it is well known that they reduce the Klein-Gordon equation into two first-order Dirac wave equations [**17**]. In view of the possible construction of an effective wave equation for the three-quark proton wave function, it is important to investigate whether there are gamma matrices acting on E.

A first example is easily found as follows. We have already noticed that, if the proton spinor wave function belongs to the subspace $E_1 = D(0, \frac{1}{2}) \oplus D(\frac{1}{2}, 0)$, it transforms like a Dirac 4-spinor. The associated spin generators are just the restrictions to E_1 of the general spin generators $\mathrm{S}_{\mu\nu}$ (2.2) acting on the entire space E :

[8] See, however, Ref. [**11**].

$$\Sigma_{\mu\nu} = \mathcal{Q}_1 \mathrm{S}_{\mu\nu} \mathcal{Q}_1 \tag{5.2}$$

$\mathcal{Q}_1$ being the projection operator onto E_1 (see Eq.2.18a). It can be shown that they display the same simple properties as $\sigma_{\mu\nu} = i[\gamma_\mu, \gamma_\nu]/4$. Indeed, it is a well-known particularity of any representation $D(0, \frac{1}{2}) \oplus D(\frac{1}{2}, 0)$ that the corresponding infinitesimal Lorentz generators are connected to gamma matrices Γ_μ by the simple relation [**9**] :

$$\Sigma_{\mu\nu} = \frac{i}{4}\left[\Gamma_\mu, \Gamma_\nu\right] \tag{5.3}$$

Explicitly, if we define :

$$\Gamma^k = 2i\Gamma_0\Sigma_{0k} \quad (k = 1, 2, 3) \tag{5.4}$$

with $\Gamma_0 = \gamma_0 \otimes \gamma_0 \otimes \gamma_0$, we then obtain a set of four gamma matrices, namely $(\Gamma_0,\ \Gamma^1,\ \Gamma^2,\ \Gamma^3)$, which operate only on E_1 and may be called the *genuine* Dirac matrices associated with the *genuine* spin 1/2 spinor wave functions of E_1.

Of course, this does not exhaust all possibilities regarding gamma matrices. In fact, reducing, with respect to the Lorentz spin group, the space $E \times E$ (which is isomorphic to the 4096-dimensional space $\mathcal{M}$ of all matrices acting on E), the 4-vector representation $D(\frac{1}{2}, \frac{1}{2})$ is obtained 140 times, which, surely, leaves a great deal of room for other gamma matrices. In particular, the matrices :

$$\lambda_\mu(x, y, z) = i\left(x\ \gamma_\mu\gamma_5 \otimes 1_4 \otimes \gamma_5 + y\ \gamma_5 \otimes \gamma_\mu\gamma_5 \otimes 1_4 + z\ 1_4 \otimes \gamma_5 \otimes \gamma_\mu\gamma_5\right) \tag{5.5}$$

where $x,\ y$ and z are any three coefficients, verify the anticommutation relations :

$$\left[\lambda_\mu(x, y, z),\ \lambda_\nu(x', y', z')\right]_+ = 2g_{\mu\nu}\left(xx' + yy' + zz'\right) \tag{5.6}$$

They may be considered as the extension to three-quark systems of the Dirac matrices found by Leutwyller and Stern [**18**] in the meson case. Considering $x,\ y$ and z as the components of a 3-vector, and choosing three orthogonal and normalized 3-vectors, we then get three independent sets of gamma matrices. It can be easily shown that the twelve gamma matrices thus derived may be used, through their successive products, so as to generate the entire space $\mathcal{M}$. However, they are not directly connected to the spin generators (2.2) by a simple formula like (5.3).

To illustrate the usefulness of gamma matrices in constructing an effective equation for the proton wave function, we shall consider the following toy model where the "unperturbed" Lagrangian density is given by :

$$\begin{aligned}\mathcal{L}_0(x_1, x_2, x_3) = \frac{i}{2}\bar{\Psi} g_\mu \partial_X^\mu \Psi - \frac{i}{2}\partial_X^\mu \bar{\Psi} g_\mu \Psi \\ +\kappa\left(\partial_\xi^\mu \bar{\Psi} M_{\mu\nu} \partial_\xi^\nu \Psi + \tfrac{3}{4}\partial_\eta^\mu \bar{\Psi} M_{\mu\nu} \partial_\eta^\nu \Psi\right)\end{aligned} \tag{5.7}$$

where $\Psi \equiv \Psi(x_1, x_2, x_3)$ is the proton wave function and $\bar{\Psi} = \Psi^\dagger \Gamma_0$; $x_1,\ x_2$ and x_3 represent the respective space-time coordinates of quark 1, 2 and 3 ; $\partial_X,\ \partial_\xi$ and ∂_η are, respectively, the partial derivatives with respect to the coordinates :

$$\begin{aligned} X &= \tfrac{1}{3}\left(x_1 + x_2 + x_3\right) \\ \xi &= x_1 - x_2 \\ \eta &= \tfrac{1}{2}\left(x_1 + x_2 - 2x_3\right) \end{aligned} \tag{5.8}$$

The parameter κ will be defined below. Finally, g_μ are gamma matrices such that $\Gamma_0 g_\mu^\dagger \Gamma_0 = g_\mu$, and $M_{\mu\nu}$ is a 4-tensor matrix verifying :

$$M_{\mu\nu} + M_{\nu\mu} = 2g_{\mu\nu} \quad , \quad \Gamma_0 M_{\mu\nu}^\dagger \Gamma_0 = M_{\nu\mu} \tag{5.9}$$

The principle of stationary action yields the equation :

$$\frac{\partial \mathcal{L}_0}{\partial \bar{\Psi}} = \sum_{r=1}^{3} \frac{\partial}{\partial x_r^\mu} \frac{\partial \mathcal{L}_0}{\partial(\partial \bar{\Psi}/\partial x_{r\mu})} \tag{5.10}$$

whence one derives the "unperturbed" wave equation :

$$i\partial_X^\mu g_\mu \Psi = \kappa \left(\partial_\xi^2 \Psi + \frac{3}{4} \partial_\eta^2 \Psi \right) \tag{5.11}$$

where $\partial^2 \equiv \partial^\mu \partial_\mu$. If we assume that the proton wave function satisfies the "overall Dirac equation" :

$$\left(i\partial_X^\mu g_\mu - M\right) \Psi = 0 \tag{5.12}$$

M being the proton mass, then, taking account of :

$$\partial_X^2 + 6\left(\partial_\xi^2 + \frac{3}{4}\partial_\eta^2\right) = 3\left(\partial_1^2 + \partial_2^2 + \partial_3^2\right) \tag{5.13}$$

with $\partial_r^\mu \equiv \partial/\partial x_r^\mu,\ r = 1,\ 2,\ 3$, we get:

$$\left(\partial_1^2 + \partial_2^2 + \partial_3^2\right) \Psi = \frac{1}{3}\left(-M^2 + \frac{6M}{\kappa}\right) \Psi \tag{5.14}$$

We thus obtain a generalized Klein-Gordon equation :

$$\left(\partial_1^2 + \partial_2^2 + \partial_3^2 + \frac{M_0^2}{3}\right) \Psi = 0 \tag{5.15}$$

provided we take :

$$\kappa = -\frac{6M}{M_0^2 - M^2} < 0 \tag{5.16}$$

Afterwards, QCD interactions are introduced in our toy model according to the usual minimal coupling prescription :

$$\partial/\partial x_{r\mu} \to \partial/\partial x_{r\mu} + ig_s \sum_a \frac{\lambda_a}{2} G_\mu^a(x_r) \tag{5.17}$$

G_μ^a being the gluon field and λ_a the Gell-Mann color matrices ; g_s is the strong coupling constant. Since effective wave equations are meant for describing confinement phenomena in the non-perturbative regime of QCD, we feel justified to here consider the gluon field as an external background colored gauge field and to approximate products of gluon fields by their vacuum expectation values, according to a well-known non-perturbative method described in Refs. [**19, 20**]. In that approximation, only the terms involving two gluon fields contribute in the "perturbed" Lagrangian. On the other hand, if we make the plausible assumption that the space-time quark coordinates are essentially concentrated around the mean value $\approx X$, then, using a fixed-point gauge representation of the gluon field [**19**, **20**], we may apply the additional approximations :

$$G_\mu^a(x_r) \approx \frac{1}{2}(x_r - X)^\nu G_{\nu\mu}^a(X) \tag{5.18}$$

where $G_{\nu\mu}^a$ is the gluon field strength tensor. We thus get, for instance :

$$G_\mu^a(x_3) G_\nu^b(x_3) \approx \frac{1}{9} \eta^\rho \eta^\sigma < 0|G_{\rho\mu}^a(X) G_{\sigma\nu}^b(X)|0 > \tag{5.19}$$

and using :

$$< 0|G_{\rho\mu}^a(X) G_{\sigma\nu}^b(X)|0 >= \frac{1}{96} \delta^{ab}(g_{\rho\sigma} g_{\mu\nu} - g_{\rho\nu} g_{\mu\sigma}) < G^2 > \tag{5.20}$$

where $< G^2 >$ is the gluon condensate, we obtain :

$$G_\mu^a(x_3) G_\nu^b(x_3) M^{\mu\nu} \approx \frac{1}{288} \delta^{ab} \eta^2 < G^2 > \tag{5.21}$$

After some algebra, the increment of the Lagrangian density is found to be :

$$\Delta\mathcal{L} = \bar{\Psi} \frac{g_s^2}{576} < G^2 > (\eta^2 + \frac{3}{4}\xi^2)\Psi \tag{5.22}$$

whence one gets the following wave equation :

$$i\partial_X^\mu \lambda_\mu \Psi = \kappa \left(\partial_\xi^2 + \frac{3}{4}\partial_\eta^2 - \frac{g_s^2}{576} < G^2 > (\eta^2 + \frac{3}{4}\xi^2) \right) \Psi \tag{5.23}$$

If we assume Eq.(5.12) to be still valid, the following picture appears : while the proton wave function would satisfy an overall equation resembling a Dirac equation with respect to the overall "barycentric coordinates" X, its mass M would be given through an "internal wave equation" :

$$M\Psi = \kappa \left(\partial_\xi^2 + \frac{3}{4}\partial_\eta^2 - \frac{g_s^2}{576} < G^2 > (\eta^2 + \frac{3}{4}\xi^2) \right) \Psi \tag{5.24}$$

Moreover, under the approximations described above, the effect of the vacuum gluon field is to produce an harmonic confinement potential. Such an interesting correlation between the gluon condensate and confining harmonic forces has already been emphasized by the authors of Ref. [**21**] who used a similar, though non-relativistic approach (in particular, their Lagrangian density involves, like ours, squares of derivatives, which, in fact, are responsible of the emergence of harmonic terms via the minimal coupling (5.17)). Notice that, if we set :

$$\kappa = \frac{6M}{M^2 - 3V_0} \quad , \quad \alpha^2 = \frac{3g_s^2 < G^2 >}{64} \tag{5.25}$$

we recover the starting equation of the relativistic harmonic-oscillator model of Ref. [**7**] :

$$\begin{array}{c} \{\partial_1^2 + \partial_2^2 + \partial_3^2 + V_0 \\ -\frac{1}{27}\alpha^2 \left[(x_1 - x_2)^2 + (x_1 - x_3)^2 + (x_2 - x_3)^2\right]\Big\} \Psi = 0 \end{array} \tag{5.26}$$

Using the standard estimation [**19**] : $\alpha_s < G^2 > /\pi \simeq (0.36)^4$ GeV^4, we find for the parameter α the value $\alpha \simeq 0.18$ GeV^2 which is only a factor of 2 below the value found by Lipes in fitting the proton magnetic form factor. Though the latter result is rather encouraging, we will not discuss further implications of our (certainly too simplistic) toy model as regards, in particular, the nucleon form factors : this would require an explicit choice for the gamma matrices λ_μ, the tensor $M_{\mu\nu}$, and the spinor wave functions to be used. In consideration of the wealth of possibilities, this is surely premature.

At this point, we would like to stress that the formalism developed in Sec.4 may be used as well for a classification, with respect to the permutation group S_3, of the various differential operators that may enter a relativistic "internal wave equation" for a three-quark system, 4-momenta being simply replaced by partial 4-derivatives. However, this requires a combined classification under both S_3 and the Lorentz spin group of all the matrices acting on E. This is another complicated task, which we leave for future investigation. Let us also mention the possibility of enlarging the search for gamma matrices to the wider spin-flavor space, thus mixing spin and flavor matrices. To give an example, the matrices :

$$\Omega_\mu^{(1)} = \frac{1}{\sqrt{3}} \{\tau_1\gamma_\mu \otimes 1_8 \otimes 1_8 + \tau_2 \otimes \gamma_\mu \otimes 1_8 + \tau_3 \otimes 1_8 \otimes \gamma_\mu\} \tag{5.27}$$

where τ_1, τ_2 and τ_3 are Pauli flavor matrices, satisfy the anticommutation rule (5.1). To end up, the least we can say is that, in the context of a relativistic equation for the proton wave function, there is certainly a lot of interesting work to do in studying the 4096-dimensional basis of matrices acting on the three-quark space E, a topic just touched on in this section.

6. CONCLUDING REMARKS

In this article we have derived the most general covariant structure of a three-quark S-state wave function, including every possible mixing of spin, momentum and flavor distributions. Three-quark systems are rather complicated objects and it is interesting and helpful to make a comparison with a similar approach to covariant wave functions of much simpler systems, such as pseudoscalar $q\bar{q}$ mesons. The corresponding result is in fact well known and amounts to the expansion of any 4×4 matrix acting on a Dirac 4-spinor space according to the basis made up by the sixteen matrices :

$$\begin{array}{c} 1_4, \quad \gamma_5, \quad (\ D(0,0)\ ) \\ \gamma_\mu, \quad \gamma_\mu\gamma_5, \quad \left(\ D(\tfrac{1}{2},\tfrac{1}{2})\ \right) \\ \sigma_{\mu\nu}, \quad (\ D(0,1) \oplus D(1,0)\ ) \end{array} \tag{6.1}$$

where the associated irreducible representations of the induced Lorentz spin group for $q\bar{q}$ systems are shown in parentheses. In that case, we find, as expected, that a *genuine* pseudoscalar spinor wave function should be associated with the matrix γ_5. That too is different from the simple extension of the $SU(6)$ non-relativistic wave function, which extension is, in terms of Dirac spinors :

$$\begin{aligned} U^\uparrow \bar{V}_c^\downarrow - U^\downarrow \bar{V}_c^\uparrow &\equiv U^\uparrow \bar{V}^\uparrow + U^\downarrow \bar{V}^\downarrow \\ &= -\left\{M + \not{P}\right\}\gamma_5 \end{aligned} \tag{6.2}$$

where V_c is the spinor conjugate of $V = \gamma_5 U$. Here, M and P are, respectively, the mass and 4-momentum of the meson. On the other hand, we have :

$$\gamma_5 = V^\uparrow \bar{U}^\uparrow + V^\downarrow \bar{U}^\downarrow - U^\uparrow \bar{V}^\uparrow - U^\downarrow \bar{V}^\downarrow \tag{6.3}$$

Long ago, Llewellyn Smith derived the most general relativistic spinor structure of the Bethe-Salpeter wave function of a $q\bar{q}$ pseudoscalar meson [**22**] :

$$\psi(k,P) = \gamma_5 \left\{A + k.PB\ \not{k} + C\ \not{P} + D(\not{k}\ \not{P} - \not{P}\ \not{k})\right\} \tag{6.4}$$

where k is the relative 4-momentum between the two constituent quarks ; A, B, C and D are even functions of $k.P$, if the composite meson is even under charge conjugation. At this point, it is worth noticing that, if we were to describe the interaction between quarks and a pseudoscalar meson in an effective-Lagrangian theory, we would represent it by the Yukawa coupling :

$$\mathcal{L}_{q\bar{q}\pi}(x) \sim i g_{q\bar{q}\pi} \bar{q}(x)\gamma_5 q(x)\pi(x) \tag{6.5}$$

The resulting (truncated) amplitude for the transition $q+\bar{q} \to \pi$ is described by the *vertex function* γ_5, which is eventually to be multiplied by a Lorentz-invariant form factor, on account of unavoidable strong vertex corrections. We may interpret that form factor as the momentum distribution of quarks inside the bound-state meson.

More generally, in the framework of the constituent-quark model, where each hadron is considered a bound state of a definite number of quarks, transition amplitudes of hadron processes are assumed to be expressible in terms of *vertex functions*, one for each quark-hadron vertex, besides quark and gauge field propagators [**4**, **11**, **23**]. A vertex function is supposed to characterize the coupling of quarks making up a given hadron or, in other words, the quark structure of that hadron. It is derived from the associated Bethe-Salpeter wave function by multiplying the latter (in the sense of matrix multiplication), and this for each quark line, by the inverse of the corresponding one-quark propagator. For example, apart from inessential factors, the meson-quark-antiquark vertex function may be defined as [**4**] :

$$\Gamma(k,P) = S^{-1}(\frac{P+k}{2})\psi(k,P)S^{-1}(\frac{k-P}{2}) \tag{6.6}$$

where $S(p)$ is the complete one-quark propagator which has the general form :

$$S(p) = a(p^2) + b(p^2)\ \not{p} \tag{6.7}$$

In consideration of the foregoing, and more particularly of Eq.(6.5), it seems natural to associate what we here call *genuine* spin 0 or spin 1/2 wave functions with *vertex functions* for low-lying spin-0 mesons and spin-1/2 baryons respectively, rather than with Bethe-Salpeter amplitudes. In this respect, let us remark that, in the meson case, a Bethe-Salpeter amplitude like :

$$\psi'(k,P) = S(\frac{P+k}{2})\gamma_5 F(k^2, P.k) S(\frac{k-P}{2}) \tag{6.8}$$

where F is a Lorentz scalar, is perfectly compatible, taking account of (6.7), with the general form (6.4) (with special coefficients). Moreover, in the weak-binding limit, the numerators of the two propagators entering (6.8) are usually reduced respectively to $\sim M+\ \not{P}$ and $\sim M-\ \not{P}$, so that, in that approximation, the Bethe-Salpeter amplitude takes on the form [**2**] :

$$\psi \sim \{M+\ \not{P}\}\,\gamma_5 \Phi(k^2, P.k) \tag{6.9}$$

which has the same spinor structure as (6.2). The last two formulas suggest the following comments.

- It appears that quark propagator factors transform the *genuine* spin-0 wave function γ_5 into a wave function of the same overall spin $J = 0$, but with some of its components in representations of the Lorentz spin group different from that of γ_5. Notice that some terms even give rise to a spherical harmonics of orbital momentum $L = 1$ with respect to the relative momentum (see (6.4)). This may be somewhat troublesome when trying to interpret the Bethe-Salpeter amplitude itself as the wave function of an S state $q\bar{q}$ system. On the other hand, the existence of components in representations $D(\frac{1}{2}, \frac{1}{2})$ and $D(0,1) \oplus D(1,0)$ causes the emergence of an extra power of the momentum transfer in computing, for instance, the meson form factors (in a way analogous to that described in the Introduction). Similar conclusions may be drawn in the nucleon case. Thus, it seems important to make a clear distinction between Bethe-Salpeter amplitudes and vertex functions, especially when computing transition amplitudes. This completes our discussion of the Introduction.

- From (6.9), as a result of the weak-binding approximation, Bethe-Salpeter wave functions take on the familiar quasi-non-relativistic form, whatever the explicit binding momentum distribution of quarks, a property already emphasized in Ref. [**2**]. Indeed, it seems that this is the result of an approximation of the quark propagators rather than a property of the vertex functions themselves, as the latter may have no non-relativistic analogues.

To conclude, we hope that the general formalism here developed will serve for further detailed investigations of wave functions (*vertex functions*) of three-quark bound states, in particular in the context of an effective relativistic wave equation for the proton, which may be an alternative non-perturbative approach in the study of three-quark confining forces [**18**].

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m_5	u	J	Representation
$+1$	-1	$1/2$	$D(\frac{1}{2},0) \oplus D(0,\frac{1}{2})$
$-5/3$	-1	$1/2$	$D(\frac{1}{2},1) \oplus D(1,\frac{1}{2})$
$1/3$	$+1$	$3/2$	
-1	$+1$	$3/2$	$D(\frac{3}{2},0) \oplus D(0,\frac{3}{2})$

Table I : Dimensions of representations $D(j_1, j_2) \oplus D(j_2, j_1)$ of the Lorentz spin group in E.

	E_1	$E_{-5/3}$	$E_{1/3}$	E_{-1}	Total
$Y_{M(A)}$	8	4	8	0	20
$Y_{M(S)}$	8	4	8	0	20
Y_A	4	0	0	0	4
Y_S	0	4	8	8	20
Total	20	12	24	8	64

Table II : Dimensions of the intersection subspaces $Y \cap E_{m_5}$.

N	d_D	d_S	d_A	$d_N = N + 1$
1	2	0	0	2
2	2	1	0	3
3	2	1	1	4
4	4	1	0	5
5	4	1	1	6
6	4	2	1	7
7	6	1	1	8
8	6	2	1	9
9	6	2	2	10
10	8	2	1	11
11	8	2	2	12
12	8	3	2	13
13	10	2	2	14
14	10	3	2	15
15	10	3	3	16

Table III : Dimensions of irreducible representations of S_3 by polynomials in $z_1 = P.p_\xi$ and $z_2 = -\frac{\sqrt{3}}{2}P.p_\eta$.

Appendix A :
Basis of third-rank eigenspinors

Here, we only give the generic spinor forms of positive parity and helicity "up". For $J = 1/2$, we have the eight basic spinors :

$$a_1 = \frac{1}{\sqrt{2}} \left(U^\uparrow U^\downarrow - U^\downarrow U^\uparrow\right) U^\uparrow \tag{A.1}$$

$$\begin{aligned} a_2 = \frac{1}{\sqrt{6}} \Big\{ & \left(V^\uparrow V^\downarrow - V^\downarrow V^\uparrow\right) U^\uparrow + \Big(U^\uparrow V^\downarrow + \\ & -U^\downarrow V^\uparrow + V^\uparrow U^\downarrow - V^\downarrow U^\uparrow\Big) V^\uparrow \Big\} \end{aligned} \tag{A.2}$$

$$\begin{aligned} a_3 = \frac{1}{\sqrt{6}} \Big\{ & - \left(V^\uparrow V^\downarrow - V^\downarrow V^\uparrow\right) U^\uparrow + \Big(V^\uparrow U^\downarrow + \\ & -U^\downarrow V^\uparrow\Big) V^\uparrow + \left(U^\uparrow V^\uparrow - V^\uparrow U^\uparrow\right) V^\downarrow \Big\} \end{aligned} \tag{A.3}$$

$$s_1 = \frac{1}{\sqrt{6}} \left\{ \left(U^\uparrow U^\downarrow + U^\downarrow U^\uparrow\right) U^\uparrow - 2 U^\uparrow U^\uparrow U^\downarrow \right\} \tag{A.4}$$

$$\begin{aligned} s_2 = \frac{1}{\sqrt{18}} \Big\{ & \left(V^\uparrow U^\downarrow + V^\downarrow U^\uparrow + U^\uparrow V^\downarrow + U^\downarrow V^\uparrow\right) V^\uparrow + \\ & -2 \left(U^\uparrow V^\uparrow + V^\uparrow U^\uparrow\right) V^\downarrow + \left(V^\uparrow V^\downarrow + V^\downarrow V^\uparrow\right) U^\uparrow - 2 V^\uparrow V^\uparrow U^\downarrow \Big\} \end{aligned} \tag{A.5}$$

$$\begin{aligned} s_3 = \frac{1}{\sqrt{18}} \Big\{ & \left(V^\uparrow U^\downarrow + U^\downarrow V^\uparrow\right) V^\uparrow + \left(U^\uparrow V^\uparrow + V^\uparrow U^\uparrow\right) V^\downarrow + \\ & -2 \left(U^\uparrow V^\downarrow + V^\downarrow U^\uparrow\right) V^\uparrow + \left(V^\uparrow V^\downarrow + V^\downarrow V^\uparrow\right) U^\uparrow - 2 V^\uparrow V^\uparrow U^\downarrow \Big\} \end{aligned} \tag{A.6}$$

$$\begin{aligned} \mathcal{S} = \frac{1}{\sqrt{18}} \Big\{ & \left(V^\uparrow V^\downarrow + V^\downarrow V^\uparrow\right) U^\uparrow + \left(U^\uparrow V^\downarrow + V^\downarrow U^\uparrow\right) V^\uparrow + \\ & + \left(V^\uparrow U^\uparrow + U^\uparrow V^\uparrow\right) V^\downarrow - 2 U^\downarrow V^\uparrow V^\uparrow - 2 V^\uparrow U^\downarrow V^\uparrow - 2 V^\uparrow V^\uparrow U^\downarrow \Big\} \end{aligned} \tag{A.7}$$

$$\begin{aligned} \mathcal{A} = \frac{1}{\sqrt{6}} \Big\{ & \left(U^\uparrow V^\downarrow - V^\downarrow U^\uparrow\right) V^\uparrow - \left(V^\uparrow V^\downarrow - V^\downarrow V^\uparrow\right) U^\uparrow + \\ & + \left(V^\uparrow U^\uparrow - U^\uparrow V^\uparrow\right) V^\downarrow \Big\} \end{aligned} \tag{A.8}$$

The eight basic spinors for $J = 3/2$ are :

$$\begin{aligned} a'_1 = \frac{1}{\sqrt{6}} \Big\{ & \left(V^\uparrow U^\uparrow - U^\uparrow V^\uparrow\right) V^\downarrow + \\ & + \left(V^\uparrow U^\downarrow - U^\uparrow V^\downarrow\right) V^\uparrow + \left(V^\downarrow U^\uparrow - U^\downarrow V^\uparrow\right) V^\uparrow \Big\} \end{aligned} \tag{A.9}$$

$$a_2' = \frac{1}{\sqrt{2}} \left(V^\uparrow U^\uparrow - U^\uparrow V^\uparrow \right) V^\uparrow \tag{A.10}$$

$$s_1' = \frac{1}{\sqrt{18}} \Big\{ \left(V^\uparrow U^\uparrow + U^\uparrow V^\uparrow \right) V^\downarrow + \left(V^\uparrow U^\downarrow + U^\uparrow V^\downarrow \right) V^\uparrow +$$
$$+ \left(V^\downarrow U^\uparrow + U^\downarrow V^\uparrow \right) V^\uparrow - 2V^\uparrow V^\uparrow U^\downarrow - 2V^\uparrow V^\downarrow U^\uparrow - 2V^\downarrow V^\uparrow U^\uparrow \Big\} \tag{A.11}$$

$$s_2' = \frac{1}{\sqrt{6}} \Big\{ \left(V^\uparrow U^\uparrow + U^\uparrow V^\uparrow \right) V^\uparrow - 2V^\uparrow V^\uparrow U^\uparrow \Big\} \tag{A.12}$$

$$\mathcal{S}_1' = \frac{1}{\sqrt{3}} \left\{ U^\uparrow U^\uparrow U^\downarrow + U^\uparrow U^\downarrow U^\uparrow + U^\downarrow U^\uparrow U^\uparrow \right\} \tag{A.13}$$

$$\mathcal{S}_1'' = U^\uparrow U^\uparrow U^\uparrow \tag{A.14}$$

$$\mathcal{S}_2' = \frac{1}{3} \Big\{ V^\uparrow V^\uparrow U^\downarrow + V^\uparrow V^\downarrow U^\uparrow + V^\downarrow V^\uparrow U^\uparrow + V^\uparrow U^\uparrow V^\downarrow +$$
$$+ V^\uparrow U^\downarrow V^\uparrow + V^\downarrow U^\uparrow V^\uparrow + U^\uparrow V^\uparrow V^\downarrow + U^\uparrow V^\downarrow V^\uparrow + U^\downarrow V^\uparrow V^\uparrow \Big\} \tag{A.15}$$

$$\mathcal{S}_2'' = \frac{1}{\sqrt{3}} \left\{ V^\uparrow V^\uparrow U^\uparrow + V^\uparrow U^\uparrow V^\uparrow + U^\uparrow V^\uparrow V^\uparrow \right\} \tag{A.16}$$

- Basis of $D(0, \frac{1}{2}) \oplus D(\frac{1}{2}, 0)$:

$\frac{1}{2}\left(a_1 + \sqrt{3}a_2\right),\ \frac{1}{2}\left(s_1 + \sqrt{3}s_2\right),\ \frac{1}{2\sqrt{2}}\left(\sqrt{3}a_1 - a_2 - 2a_3\right),\ \frac{1}{2\sqrt{2}}\left(\sqrt{3}s_1 - s_2 - 2s_3\right),\ \mathcal{A}$

- Basis of $D(1, \frac{1}{2}) \oplus D(\frac{1}{2}, 1)$:

$\frac{1}{2\sqrt{2}}\left(\sqrt{3}a_1 - a_2 + 2a_3\right),\ \frac{1}{2\sqrt{2}}\left(\sqrt{3}s_1 - s_2 + 2s_3\right),\ \mathcal{S},\ a_1',\ s_1',\ a_2',\ s_2'$

$\frac{1}{2}\left(\sqrt{3}\mathcal{S}_1' - \mathcal{S}_2'\right),\ \frac{1}{2}\left(\sqrt{3}\mathcal{S}_1'' - \mathcal{S}_2''\right)$

- Basis of $D(0, \frac{3}{2}) \oplus D(\frac{3}{2}, 0)$:

$\frac{1}{2}\left(\mathcal{S}_1' + \sqrt{3}\mathcal{S}_2'\right),\ \frac{1}{2}\left(\mathcal{S}_1'' + \sqrt{3}\mathcal{S}_2''\right),$

Appendix B :

Generalized Rarita-Schwinger tensor-spinors[8]

The matrix $A = \mathcal{C}\gamma_5 = -{}^T A$ transforms a Dirac spinor U_σ of helicity σ into

$$A_{\alpha\beta} \left(\bar{U}_\sigma\right)_\beta = (-1)^{\frac{1}{2}+\sigma} \left(U_{-\sigma}\right)_\alpha \tag{B.1}$$

where implicit summation on β is assumed. We thus have (with $V = \gamma_5 U$)

$$\begin{aligned} (U^\downarrow)_\alpha &= (\bar{U}^\uparrow)_\beta \, A_{\beta\alpha} \ , \quad (U^\uparrow)_\alpha = -(\bar{U}^\downarrow)_\beta \, A_{\beta\alpha} \\ (V^\downarrow)_\alpha &= -(\bar{V}^\uparrow)_\beta \, A_{\beta\alpha} \ , \quad (V^\uparrow)_\alpha = (\bar{V}^\downarrow)_\beta \, A_{\beta\alpha} \end{aligned} \tag{B.2}$$

Let x, y and z be three orthogonal and spacelike 4-vectors which are also orthogonal to P. Defining the helicities of spinors as spin projections on z, we have (see Ref. [**5**])

$$\begin{aligned} U^{\uparrow,\downarrow}\bar{U}^{\uparrow,\downarrow} &= \frac{1}{2}\left(1 \pm \gamma_5 \not{z}\right)\left(M + \not{P}\right) \\ U^{\uparrow,\downarrow}\bar{U}^{\downarrow,\uparrow} &= \frac{1}{2}\gamma_5\left(\not{x} \pm i \not{y}\right)\left(M + \not{P}\right) \end{aligned} \tag{B.3}$$

where Dirac spinors are here normalized according to $\bar{U}U = -\bar{V}V = 2M$. These simple relations, together with the well-known closure relations

$$\begin{aligned} U^\uparrow\bar{U}^\uparrow + U^\downarrow\bar{U}^\downarrow &= M + \not{P} \\ V^\uparrow\bar{V}^\uparrow + V^\downarrow\bar{V}^\downarrow &= -M + \not{P} \end{aligned} \tag{B.4}$$

allow us to express the elements of any 4×4 matrix in terms of second-rank spinors UU, UV, VU and VV. In particular, using simplified notations for the tensor products, we have the following expansions

$$A = \frac{1}{2M}\left\{U^\uparrow U^\downarrow - U^\downarrow U^\uparrow + V^\uparrow V^\downarrow - V^\downarrow V^\uparrow\right\} \tag{B.5}$$

$$\not{P}A = \frac{1}{2}\left\{U^\uparrow U^\downarrow - U^\downarrow U^\uparrow - V^\uparrow V^\downarrow + V^\downarrow V^\uparrow\right\} \tag{B.6}$$

$$\gamma_5 A = \frac{1}{2M}\left\{V^\uparrow U^\downarrow - V^\downarrow U^\uparrow + U^\uparrow V^\downarrow - U^\downarrow V^\uparrow\right\} \tag{B.7}$$

$$\gamma_5 \not{P}A = \frac{1}{2}\left\{V^\uparrow U^\downarrow - V^\downarrow U^\uparrow - U^\uparrow V^\downarrow + U^\downarrow V^\uparrow\right\} \tag{B.8}$$

We then get

$$\left(M + \not{P}\right) A = U^\uparrow U^\downarrow - U^\downarrow U^\uparrow \tag{B.9}$$

From (B.3) we obtain

$$\bar{U}^{\uparrow,\downarrow}\gamma_\mu U^{\uparrow,\downarrow} = 2P_\mu \ , \quad \bar{U}^{\uparrow,\downarrow}\gamma_\mu U^{\downarrow,\uparrow} = 0$$
$$\bar{U}^{\uparrow,\downarrow}\gamma_\mu V^{\uparrow,\downarrow} = \pm 2M z_\mu \ , \quad \bar{U}^{\uparrow,\downarrow}\gamma_\mu V^{\downarrow,\uparrow} = 2M\,(x \mp iy)_\mu \tag{B.10}$$

whence we derive the formulas

$$\begin{aligned} 2M\gamma_\mu = {} & t_\mu \left\{U^\uparrow\bar{U}^\uparrow + U^\downarrow\bar{U}^\downarrow + V^\uparrow\bar{V}^\uparrow + V^\downarrow\bar{V}^\downarrow\right\} + \\ & -z_\mu \left\{U^\uparrow\bar{V}^\uparrow - U^\downarrow\bar{V}^\downarrow + V^\uparrow\bar{U}^\uparrow - V^\downarrow\bar{U}^\downarrow\right\} \\ & +\sqrt{2}\, e^{(+)}_\mu \left\{V^\downarrow\bar{U}^\uparrow + U^\downarrow\bar{V}^\uparrow\right\} - \sqrt{2}\, e^{(-)}_\mu \left\{U^\uparrow\bar{V}^\downarrow + V^\uparrow\bar{U}^\downarrow\right\} \end{aligned} \tag{B.11}$$

$$\begin{aligned} 2M\gamma_\mu A = {} & t_\mu \left\{U^\uparrow U^\downarrow - U^\downarrow U^\uparrow - V^\uparrow V^\downarrow + V^\downarrow V^\uparrow\right\} + \\ & -z_\mu \left\{V^\uparrow U^\downarrow + V^\downarrow U^\uparrow - U^\uparrow V^\downarrow - U^\downarrow V^\uparrow\right\} \\ & +\sqrt{2}\, e^{(+)}_\mu \left\{V^\downarrow U^\downarrow - U^\downarrow V^\downarrow\right\} - \sqrt{2}\, e^{(-)}_\mu \left\{U^\uparrow V^\uparrow - V^\uparrow U^\uparrow\right\} \end{aligned} \tag{B.12}$$

and

$$\gamma_\mu U^\uparrow = t_\mu U^\uparrow - z_\mu V^\uparrow + \sqrt{2}\, e^{(+)}_\mu V^\downarrow \tag{B.13}$$

Here, for simplicity, we have set $t_\mu = P_\mu/M$, $e^{(\pm)} = \mp\frac{1}{\sqrt{2}}(x \pm iy)_\mu$. We thus get

$$\begin{aligned} \not{e}^{(+)} A &= -\frac{1}{M\sqrt{2}}\left\{U^\uparrow V^\uparrow - V^\uparrow U^\uparrow\right\} \\ \not{e}^{(-)} A &= -\frac{1}{M\sqrt{2}}\left\{U^\downarrow V^\downarrow - V^\downarrow U^\downarrow\right\} \end{aligned} \tag{B.14}$$

From (B.11) we obtain

$$\begin{aligned} M[\gamma_\mu\,,\,\gamma_\nu] = {} & -\left\{t_\mu z_\nu - t_\nu z_\mu\right\}\left\{U^\uparrow\bar{V}^\uparrow - U^\downarrow\bar{V}^\downarrow - V^\uparrow\bar{U}^\uparrow + V^\downarrow\bar{U}^\downarrow\right\} + \\ & -\sqrt{2}\left\{t_\mu e^{(+)}_\nu - t_\nu e^{(+)}_\mu\right\}\left\{V^\downarrow\bar{U}^\uparrow - U^\downarrow\bar{V}^\uparrow\right\} + \\ & +\sqrt{2}\left\{t_\mu e^{(-)}_\nu - t_\nu e^{(-)}_\mu\right\}\left\{V^\uparrow\bar{U}^\downarrow - U^\uparrow\bar{V}^\downarrow\right\} + \\ & -\sqrt{2}\left\{z_\mu e^{(+)}_\nu - z_\nu e^{(+)}_\mu\right\}\left\{U^\downarrow\bar{U}^\uparrow - V^\downarrow\bar{V}^\uparrow\right\} + \\ & +\sqrt{2}\left\{z_\mu e^{(-)}_\nu - z_\nu e^{(-)}_\mu\right\}\left\{V^\uparrow\bar{V}^\downarrow - U^\uparrow\bar{U}^\downarrow\right\} + \\ & -\left\{e^{(+)}_\mu e^{(-)}_\nu - e^{(+)}_\nu e^{(-)}_\mu\right\}\left\{U^\uparrow\bar{U}^\uparrow + V^\downarrow\bar{V}^\downarrow - U^\downarrow\bar{U}^\downarrow - V^\uparrow\bar{V}^\uparrow\right\} \end{aligned} \tag{B.15}$$

Whence

$$\frac{1}{2}[\gamma_\mu\,,\,\gamma_\nu]\,U^\uparrow = \{t_\mu z_\nu - t_\nu z_\mu\}\,V^\uparrow - \left\{e_\mu^{(+)}e_\nu^{(-)} - e_\nu^{(+)}e_\mu^{(-)}\right\}U^\uparrow +$$
$$-\sqrt{2}\,\left\{t_\mu e_\nu^{(+)} - t_\nu e_\mu^{(+)}\right\}V^\downarrow - \sqrt{2}\,\left\{z_\mu e_\nu^{(+)} - z_\nu e_\mu^{(+)}\right\}U^\downarrow \tag{B.16}$$

and

$$[\gamma_\mu\,,\,\not{P}] = z_\mu\left\{U^\uparrow\bar{V}^\uparrow - U^\downarrow\bar{V}^\downarrow - V^\uparrow\bar{U}^\uparrow + V^\downarrow\bar{U}^\downarrow\right\} +$$
$$+\sqrt{2}\,e_\mu^{(+)}\left\{V^\downarrow\bar{U}^\uparrow - U^\downarrow\bar{V}^\uparrow\right\} - \sqrt{2}\,e_\mu^{(-)}\left\{V^\uparrow\bar{U}^\downarrow - U^\uparrow\bar{V}^\downarrow\right\} \tag{B.17}$$

Afterwards, it is easy to derive the following formulas

$$2M\,(\gamma_\mu A)_{\alpha\beta}\left(\gamma_\mu U^\uparrow\right)_\delta = \left\{U^\uparrow U^\downarrow - U^\downarrow U^\uparrow - V^\uparrow V^\downarrow + V^\downarrow V^\uparrow\right\}_{\alpha\beta}\,U_\delta^\uparrow$$
$$-\left\{V^\uparrow U^\downarrow - U^\downarrow V^\uparrow - U^\uparrow V^\downarrow + V^\downarrow U^\uparrow\right\}_{\alpha\beta}\,V_\delta^\uparrow + \tag{B.18}$$
$$+2\left\{V^\uparrow U^\uparrow - U^\uparrow V^\uparrow\right\}_{\alpha\beta}\,V_\delta^\downarrow$$

$$\{\bar{\sigma}_{\mu\nu}P^\nu A\}_{\alpha\beta}\left(\gamma^\mu\,U^\uparrow\right)_\delta = \left\{U^\uparrow V^\uparrow + V^\uparrow U^\uparrow\right\}_{\alpha\beta}\,V_\delta^\downarrow +$$
$$-\frac{1}{2}\left\{U^\uparrow V^\downarrow + U^\downarrow V^\uparrow + V^\uparrow U^\downarrow + V^\downarrow U^\uparrow\right\}_{\alpha\beta}\,V_\delta^\uparrow \tag{B.19}$$

$$\{\gamma_5\bar{\sigma}_{\mu\nu}P^\nu A\}_{\alpha\beta}\left(\gamma_5\gamma^\mu\,U^\uparrow\right)_\delta = \left\{V^\uparrow V^\uparrow + U^\uparrow U^\uparrow\right\}_{\alpha\beta}\,U_\delta^\downarrow +$$
$$-\frac{1}{2}\left\{U^\uparrow U^\downarrow + U^\downarrow U^\uparrow + V^\uparrow V^\downarrow + V^\downarrow V^\uparrow\right\}_{\alpha\beta}\,U_\delta^\uparrow \tag{B.20}$$

$$(\gamma_\mu A)_{\alpha\beta}\left\{\bar{\sigma}_{\mu\nu}P^\nu\,U^\uparrow\right\}_\delta = \left\{V^\uparrow U^\uparrow - U^\uparrow V^\uparrow\right\}_{\alpha\beta}\,V_\delta^\downarrow +$$
$$+\frac{1}{2}\left\{U^\uparrow V^\downarrow + U^\downarrow V^\uparrow - V^\uparrow U^\downarrow - V^\downarrow U^\uparrow\right\}_{\alpha\beta}\,V_\delta^\uparrow \tag{B.21}$$

$$(\gamma_5\gamma_\mu A)_{\alpha\beta}\left\{\gamma_5\bar{\sigma}_{\mu\nu}P^\nu\,U^\uparrow\right\}_\delta = \left\{U^\uparrow U^\uparrow - V^\uparrow V^\uparrow\right\}_{\alpha\beta}\,U_\delta^\downarrow +$$
$$+\frac{1}{2}\left\{V^\uparrow V^\downarrow + V^\downarrow V^\uparrow - U^\uparrow U^\downarrow - U^\downarrow U^\uparrow\right\}_{\alpha\beta}\,U_\delta^\uparrow \tag{B.22}$$

$$-M\,\{\bar{\sigma}_{\mu\nu}A\}_{\alpha\beta}\left\{\bar{\sigma}^{\mu\nu}U^\uparrow\right\}_\delta = \left\{U^\uparrow U^\downarrow + U^\downarrow U^\uparrow + V^\uparrow V^\downarrow + V^\downarrow V^\uparrow\right\}_{\alpha\beta}\,U_\delta^\uparrow +$$
$$+\left\{U^\uparrow V^\downarrow + U^\downarrow V^\uparrow + V^\uparrow U^\downarrow + V^\downarrow U^\uparrow\right\}_{\alpha\beta}\,V_\delta^\uparrow + \tag{B.23}$$
$$-2\left\{U^\uparrow U^\uparrow + V^\uparrow V^\uparrow\right\}_{\alpha\beta}\,U_\delta^\downarrow - 2\left\{V^\uparrow U^\uparrow + U^\uparrow V^\uparrow\right\}_{\alpha\beta}\,V_\delta^\downarrow$$

where we have defined $\bar{\sigma}_{\mu\nu} = [\gamma_\mu\,,\,\gamma_\nu]/2$.

With all this formulas at hand, we may rewrite the eigenspinors defined in Appendix A by means of generalized Rarita-Schwinger functions, wich have the form $(\Gamma A)_{\alpha\beta}(\Gamma' U)_\delta$, where Γ and Γ' are 4×4 matrices constructed from the 16 Dirac matrices 1, γ_5, γ_μ, $\gamma_5\gamma_\mu$ and $\bar\sigma_{\mu\nu}$. For instance, for $J=1/2$ spinors, we have (apart from a normalization factor)

$$a_1 = \frac{1}{\sqrt{2}}\left\{(M+\not P)\,A\right\}_{\alpha\beta}\,U^{\uparrow}_{\delta} \tag{B.24}$$

$$s_1 = \frac{1}{\sqrt{6}}\left\{(\gamma_5\gamma^\nu A)_{\alpha\beta}\left(\gamma_5\bar\sigma_{\mu\nu}P^\mu U^\uparrow\right)_\delta + (\gamma_5\bar\sigma_{\mu\nu}P^\mu A)\left(\gamma_5\gamma^\mu U^\uparrow\right)_\delta\right\} \tag{B.25}$$

$$\frac{1}{2}\left\{a_1+\sqrt{3}a_2\right\} = \frac{M}{\sqrt{2}}\left\{A_{\alpha\beta}\,U^\uparrow_\delta + (\gamma_5 A)_{\alpha\beta}\left(\gamma_5 U^\uparrow\right)_\delta\right\} \tag{B.26}$$

$$\frac{1}{2}\left\{s_1+\sqrt{3}s_2\right\} = -\frac{M}{2\sqrt{6}}\,(\bar\sigma_{\mu\nu}A)_{\alpha\beta}\left(\bar\sigma^{\mu\nu}U^\uparrow\right)_\delta \tag{B.27}$$

$$\mathcal{A} = \frac{M}{\sqrt{6}}\left\{(\gamma^\mu A)_{\alpha\beta}\left(\gamma_\mu U^\uparrow\right)_\delta - A_{\alpha\beta}\,U^\uparrow_\delta + (\gamma_5 A)_{\alpha\beta}\left(\gamma_5 U^\uparrow\right)_\delta\right\} \tag{B.28}$$

$$\mathcal{S} = \frac{2}{\sqrt{3}}\left\{(\bar\sigma_{\mu\nu}P^\nu A)_{\alpha\beta}\left(\gamma^\mu U^\uparrow\right)_\delta + (\gamma^\mu A\gamma_5)_{\alpha\beta}\left(\gamma_5\bar\sigma_{\mu\nu}P^\nu U^\uparrow\right)_\delta\right\} \tag{B.29}$$

$$\frac{1}{2\sqrt{2}}\left\{\sqrt{3}a_1-a_2-2a_3\right\} = \frac{M}{2\sqrt{3}}\left\{2A_{\alpha\beta}\,U^\uparrow_\delta - 2\,(\gamma_5 A)_{\alpha\beta}\left(\gamma_5 U^\uparrow\right)_\delta + \right.$$
$$\left. + (\gamma_\mu A)_{\alpha\beta}\left(\gamma^\mu U^\uparrow\right)_\delta\right\} \tag{B.30}$$

$$\frac{1}{2\sqrt{2}}\left\{\sqrt{3}s_1-s_2-2s_3\right\} = -\frac{M}{2}\,(\gamma_5\gamma_\mu A)_{\alpha\beta}\left(\gamma_5\gamma^\mu U^\uparrow\right)_\delta \tag{B.31}$$

$$\frac{1}{2\sqrt{2}}\left\{\sqrt{3}a_1-a_2+2a_3\right\} = \frac{1}{2\sqrt{3}}\left\{4\,(\not P A)_{\alpha\beta}\,U^\uparrow_\delta - M\,(\gamma_\mu A)_{\alpha\beta}\left(\gamma^\mu U^\uparrow\right)_\delta\right\} \tag{B.32}$$

$$\frac{1}{2\sqrt{2}}\left\{\sqrt{3}s_1-s_2+2s_3\right\} = \frac{M}{2}\,(\gamma_5\gamma_\mu A)_{\alpha\beta}\left(\gamma_5\gamma^\mu U^\uparrow\right)_\delta +$$
$$+\frac{M}{6}\,(\bar\sigma_{\mu\nu}A)_{\alpha\beta}\left(\bar\sigma^{\mu\nu}U^\uparrow\right)_\delta + \frac{2}{3}\left\{(\gamma_5\gamma^\nu A)_{\alpha\beta}\left(\gamma_5\bar\sigma_{\mu\nu}P^\mu U^\uparrow\right)_\delta + \right. \tag{B.33}$$
$$\left. + (\gamma_5\bar\sigma_{\mu\nu}P^\mu A)_{\alpha\beta}\left(\gamma_5\gamma^\nu U^\uparrow\right)_\delta\right\}$$

The two doublets defined by (B-26)-(B.27) and by (B.30)-(B.31) respectively, as also the antisymmetric singlet (B.29), are just those derived by the authors of Ref. [**8**] from what they call spinor covariants of zero rank with respect to Lorentz indices (see formulas A.2 and A.3 in their Appendix 3).

Of course, the number of independent Rarita-Schwinger functions is limited : as third-rank spinors, these functions can take on, at most, 64 different forms. In order to illustrate this fact, let us mention the relation

$$\begin{aligned}(\gamma_5\bar{\sigma}_{\mu\nu}P^{\mu}A)_{\alpha\beta}\left(\gamma_5\gamma^{\nu}U^{\uparrow}\right)_{\delta}+(\bar{\sigma}_{\mu\nu}P^{\mu}A)_{\alpha\beta}\left(\gamma^{\nu}U^{\uparrow}\right)_{\delta}\\ =-\frac{M}{2}\left(\bar{\sigma}_{\mu\nu}A\right)_{\alpha\beta}\left(\bar{\sigma}^{\mu\nu}U^{\uparrow}\right)_{\delta}\end{aligned} \tag{B.34}$$

The limitation is even more restrictive for Rarita-Schwinger functions of a given $\mathcal{S}_3$ symmetry. In particular, we have seen that there are only four third-rank spinors which are antisymmetric singlets under $\mathcal{S}_3$. They are $\mathcal{A}^{\uparrow}\equiv\mathcal{A}$, $\mathcal{A}^{\downarrow}\equiv\mathcal{A}(\uparrow\leftrightarrow\downarrow)$, $\mathcal{A}'^{\uparrow}=\Gamma_5\mathcal{A}^{\uparrow}$, and $\mathcal{A}'^{\downarrow}=\Gamma_5\mathcal{A}^{\downarrow}$. Thus, all the antisymmetric singlets given in Appendix 3 of Ref. [**8**] are expressible in terms of these four basis forms. For example, using the notation of that reference, we have (apart from a normalisation factor)

$$\begin{aligned}\frac{M}{\sqrt{6}}\left(A_1-A_2+A_4+A_6\right)^{(\uparrow)}=t_{\mu}\mathcal{A}^{\uparrow}+\\ -z_{\mu}\mathcal{A}'^{\uparrow}-\sqrt{2}e_{\mu}^{(+)}\mathcal{A}'^{\downarrow}\end{aligned} \tag{B.35}$$

and

$$\begin{aligned}\frac{M}{\sqrt{6}}\left(B_4-B_7+B_9-B_{12}\right)^{(\uparrow)}=-\mathcal{A}'^{\uparrow}\left\{t_{\mu}z_{\nu}-t_{\nu}z_{\mu}\right\}+\\ -\mathcal{A}'^{\downarrow}\sqrt{2}\left\{t_{\mu}e_{\nu}^{(+)}-t_{\nu}e_{\mu}^{(+)}\right\}-\mathcal{A}^{\downarrow}\sqrt{2}\left\{z_{\mu}e_{\nu}^{(+)}-z_{\nu}e_{\mu}^{(+)}\right\}+\\ +\mathcal{A}^{\uparrow}\left\{e_{\mu}^{(+)}e_{\nu}^{(-)}-e_{\nu}^{(+)}e_{\mu}^{(-)}\right\}\end{aligned} \tag{B.36}$$

Appendix C :

$\mathcal{S}_3$-Irreducible momentum-distribution amplitudes

C-I. $\mathcal{S}_3$-Singlets

Let $x = z_2$, $y = z_1$ and consider the complex number $c = x + iy$. Under the permutations of $\mathcal{S}_3$, c is transformed into :

$$\begin{aligned}
\mathcal{P}_{12}\, c &= x - iy = c^\star \\
\mathcal{P}_{13}\, c &= \frac{1}{2}\left(-x + \sqrt{3}\, y\right) + \frac{i}{2}\left(y + \sqrt{3}\, x\right) = e^{i\frac{2\pi}{3}} c^\star = \left(e^{i\frac{\pi}{3}}\, c\right)^\star \\
\mathcal{P}_{12}\mathcal{P}_{13}\, c &= \frac{1}{2}\left(-x - \sqrt{3}\, y\right) + \frac{i}{2}\left(-y + \sqrt{3}\, x\right) = e^{i\frac{2\pi}{3}}\, c \\
\mathcal{P}_{13}\mathcal{P}_{12}\, c &= \frac{1}{2}\left(-x + \sqrt{3}\, y\right) - \frac{i}{2}\left(y + \sqrt{3}\, x\right) = e^{i\frac{4\pi}{3}}\, c \\
\mathcal{P}_{23}\, c &= \mathcal{P}_{12}\mathcal{P}_{13}\mathcal{P}_{12}\, c = e^{i\frac{4\pi}{3}}\, c^\star = \left(e^{i\frac{2\pi}{3}}\, c\right)^\star
\end{aligned} \tag{C.1}$$

Consider a function $f(z_1, z_2)$ which takes on real values and is a symmetric singlet under $\mathcal{S}_3$. That function may as well be expressed as a function $F(c)$ of the complex number c. Then, from Eqs.(C.1), it appears that $F(u)$ takes on the same value whenever the complex number u is one of the six roots of the algebraic equation

$$\left(u^3 - c^3\right)\left(u^3 - c^{\star 3}\right) = 0 \tag{C.2}$$

Consequently, $F(c) = f(z_1, z_2)$ should depend only on the coefficients of that equation, namely :

$$\begin{aligned}
c^3\, c^{\star 3} &= |c|^6 = \left(z_1^2 + z_2^2\right)^3 = w_1^6 \\
c^3 + c^{\star 3} &= 2|c|^3 \cos(3\theta) \equiv 2z_2\left(z_2^2 - 3z_1^2\right) = 2w_2
\end{aligned} \tag{C.3}$$

where we have set $\theta = \arg(c)$. Hence we conclude that any symmetric-singlet function $f(z_1, z_2)$ is expressible as a function of only the two independent and fundamental symmetric-singlet variables w_1 and w_2[**24**].

On the other hand, the only antisymmetric combination of c^3 and $c^{\star 3}$ is

$$\frac{1}{2}\left(c^3 - c^{\star 3}\right) = |c|^3 \sin(3\theta) \equiv -z_1\left(z_1^2 - 3z_3^2\right) = -\tilde{\mathcal{A}} \tag{C.4}$$

so that any regular antisymmetric-singlet function should be expressed as the product of $\tilde{\mathcal{A}}$ by a (regular) symmetric-singlet function as defined above.

C-II. Polynomial representations

Below are listed the irreducible representations of $\mathcal{S}_3$ by polynomials in z_1 and z_2, up to the weight $N = 6$. Doublets, symmetric singlets and antisymmetric singlets are respectively denoted by $\tilde{\mathcal{D}}$, $\tilde{\mathcal{S}}$ and $\tilde{\mathcal{A}}$; as above, we set $z_2 = \sqrt{w_1}\cos\theta$ and $z_1 = \sqrt{w_1}\sin\theta$.

1°) $N = 1$

$$\tilde{\mathcal{D}}^{(1)} \equiv \tilde{\mathcal{D}}_1 = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \sqrt{w_1}\begin{pmatrix} \sin\theta \\ \cos\theta \end{pmatrix} \tag{C.5a}$$

$$\tilde{\mathcal{S}}^{(1)} \equiv 0 \ , \quad \tilde{\mathcal{A}}^{(1)} \equiv 0 \tag{C.5b}$$

2°) $N = 2$

$$\tilde{\mathcal{D}}^{(2)} \equiv \tilde{\mathcal{D}}_2 = \begin{pmatrix} 2z_1z_2 \\ z_1^2 - z_2^2 \end{pmatrix} = w_1\begin{pmatrix} \sin(2\theta) \\ -\cos(2\theta) \end{pmatrix} \tag{C.6a}$$

$$\tilde{\mathcal{S}}^{(2)} = z_1^2 + z_2^2 \equiv w_1 \tag{C.6b}$$

$$\tilde{\mathcal{A}}^{(2)} \equiv 0 \tag{C.6c}$$

3°) $N = 3$

$$\tilde{\mathcal{D}}^{(3)} = w_1\tilde{\mathcal{D}}_1 \tag{C.7a}$$

$$\tilde{\mathcal{S}}^{(3)} = z_2(z_2^2 - 3z_1^2) = (w_1)^{3/2}\cos(3\theta) \equiv w_2 \tag{C.7b}$$

$$\tilde{\mathcal{A}}^{(3)} \equiv \tilde{\mathcal{A}} = z_1(z_1^2 - 3z_2^2) = -(w_1)^{3/2}\sin(3\theta) \tag{C.7c}$$

4°) $N = 4$

$$\tilde{\mathcal{D}}_1^{(4)} = w_2\tilde{\mathcal{D}}_1 \ , \quad \tilde{\mathcal{D}}_2^{(4)} = w_1\tilde{\mathcal{D}}_2 \tag{C.8a}$$

$$\tilde{\mathcal{S}}^{(4)} = w_1^2 \tag{C.8b}$$

$$\tilde{\mathcal{A}}^{(4)} \equiv 0 \tag{C.8c}$$

5°) $N = 5$

$$\tilde{\mathcal{D}}_1^{(5)} = w_1^2\tilde{\mathcal{D}}_1 \ , \quad \tilde{\mathcal{D}}_2^{(5)} = w_2\tilde{\mathcal{D}}_2 \tag{C.9a}$$

$$\tilde{\mathcal{S}}^{(5)} = w_1w_2 \tag{C.9b}$$

$$\tilde{\mathcal{A}}^{(5)} = w_1\tilde{\mathcal{A}} \tag{C.9c}$$

6°) $N = 6$

$$\tilde{\mathcal{D}}_1^{(6)} = w_1w_2\tilde{\mathcal{D}}_1 \ , \quad \tilde{\mathcal{D}}_2^{(6)} = w_1^2\tilde{\mathcal{D}}_2 \tag{C.10a}$$

$$\tilde{\mathcal{S}}_1^{(6)} = w_1^3 \ , \quad \tilde{\mathcal{S}}_2^{(6)} = w_2^2 \tag{C.10b}$$

$$\tilde{\mathcal{A}}^{(6)} = w_2\tilde{\mathcal{A}} \tag{C.10c}$$

C-III. Symmetric-singlet functions $f(z_1, z_2, z'_1, z'_2)$

Define the two complex numbers $c = z_2 + iz_1$ and $c' = z'_2 + iz'_1$. Both transform in the same way under $\mathcal{S}_3$, according to Eqs(C.1). Consider a real-valued function $f(z_1, z_2, z'_1, z'_2)$ which is a symmetric singlet under $\mathcal{S}_3$. We may express it as a function $F(c, c')$ of c and c'. Define also the affine function

$$\phi(x, y, c, c') = xc + yc' \tag{C.11}$$

of two real independent variables x and y. Notice that c and c' are uniquely determined by $\phi(x, y, c, c')$ (i.e. $c = \phi(1, 0, c, c')$ and $c' = \phi(0, 1, c, c')$). We may consider as well $F(c, c')$ as a functional $\mathcal{F}[\phi(x, y, c, c')]$ which, actually, does not depend neither on x nor on y. Being a symmetric singlet, $\mathcal{F}[\phi(x, y, c, c')]$ takes on the same value whenever $\phi \equiv \phi(x, y, c, c')$ is one of the six roots of the algebraic equation

$$\left(\phi^3 - (xc + yc')^3\right)\left(\phi^3 - (xc^\star + yc'^\star)^3\right) = 0 \tag{C.12}$$

Consequently, $\mathcal{F}[\phi(x, y, c, c')]$ should depend only on the coefficients of the latter equation, which are

$$K_1 = (xc + yc')^3 + (xc^\star + yc'^\star)^3 = x^3(c^3 + c^{\star 3}) +$$
$$+3x^2y(c^2c' + c^{\star 2}c'^\star) + 3xy^2(cc'^2 + c^\star c'^{\star 2}) + y^3(c'^3 + c'^{\star 3}) \tag{C.13a}$$

$$K_2 = \left\{x^2|c|^2 + xy(cc'^\star + c^\star c') + y^2|c'|^2\right\}^3 \tag{C.13b}$$

Moreover, since that functional is independent of both x and y, it must depend on the polynomials $K_1(x, y)$ and $K_2(x, y)$ only through their respective coefficients.

We may thus conclude that any symmetric-singlet $f(z_1, z_2, z'_1, z'_2)$ should be completely expressible in terms of the following basic symmetric-singlet variables

$$|c|^2 = w_1\ , \quad |c'|^2 = w'_1 \tag{C.14a}$$

$$\frac{1}{2}(c^3 + c^{\star 3}) = |c|^3 \cos(3\theta) = w_2 \tag{C.14b}$$

$$\frac{1}{2}(c'^3 + c'^{\star 3}) = |c'|^3 \cos(3\theta') = w'_2 \tag{C.14c}$$

$$\frac{1}{2}(c^2c' + c^{\star 2}c'^\star) = |c|^2|c'| \cos(2\theta + \theta') = -w_{21} \tag{C.14d}$$

$$\frac{1}{2}(c'^2c + c'^{\star 2}c^\star) = |c'|^2|c| \cos(2\theta' + \theta) = -w_{12} \tag{C.14e}$$

$$\frac{1}{2}(cc'^\star + c^\star c') = |c||c'| \cos(\theta - \theta') = w_{11} \tag{C.14f}$$

where we have set $\theta' = \arg(c')$.

As we have seen in Sec.4, w_{12} and w_{21} are rationnaly dependent on w_1, w_1' w_2, w_2' and w_{11}. Thus, only the last five forms constitute the fundamental variables from which any symmetric singlet $f(z_1, z_2, z_1', z_2')$ should be constructed.