On the maximally flatness group delay property of Bessel polynomials

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ABSTRACT

In this note, we report a simple proof that Bessel polynomials satisfy maximally flat group delay requirement for low-pass filters.

Keywords : Analog Filters, Group Delay, Bessel Polynomials

The property of maximally flat variation of group delay with respect to pulsation for a low-pass filter if its transfer function is constructed with a Bessel polynomial, is commonly derived using approximate expansions of trigonometric functions (see Ref [**1**] and references therein). In a radically different way, the approach presented here allows one to obtain the result without any expansion of function.

The proof is as follows. For an harmonic regime with pulsation ω, let us denote $\mathcal{P}(j\omega) = 1/H(j\omega)$ ($j^2 = -1$) the polynomial of degree n in the variable $j\omega$, that appears in the denominator of the transfer function $H(j\omega)$ of a low-pass filter of order n. Let us split it into its real part $U(\omega)$ and its imaginary part $V(\omega)$ and write

$$\mathcal{P}(j\omega) = |\mathcal{P}(j\omega)|\, e^{j\psi(\omega)} = U(\omega) + jV(\omega) \qquad (1)$$

so as

$$|\mathcal{P}(j\omega)|^2 = U^2(\omega) + V^2(\omega), \quad \psi(\omega) = \tan^{-1}\frac{V(\omega)}{U(\omega)} \qquad (2)$$

Assuming, as usual, that $\mathcal{P}(j\omega)$ has real coefficients, $U(\omega)$ and $V(\omega)$ are polynomials in ω that are, respectively, even and odd. Consequently, if n is even, the respective degrees of U and V are n and $n-1$. If n is odd, they are $n-1$ and n, respectively.

To express the group delay $\tau(\omega) = \dfrac{d\psi}{d\omega}(\omega)$ requires the knowledge of the derivative of the phase shift $\psi(\omega)$ with respect to ω, which we express in an obvious way by means of the derivative of $\tan\psi$:

$$\frac{d\psi}{d\omega} = \frac{1}{1+\tan^2\psi}\frac{d\tan\psi}{d\omega} = \frac{1}{U^2+V^2}\left[U\frac{dV}{d\omega} - V\frac{dU}{d\omega}\right] \qquad (3)$$

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The numerator and the denominator of the last expression are both polynomials in the variable ω^2. As polynomials in ω, the numerator has degree $2n-2$, whereas the denominator has degree $2n$, whatever the parity of n is. Hence, the impossibility to obtain a strictly constant group delay by means of polynomials, a well-known fact. However, we can approach a quasi constant value of τ in a large part of the passband $0 \le \omega \le \omega_c$, if we demand that in (3) the numerator be proportional to the denominator, up to a term proportional to $(\omega/\omega_c)^{2n}$ which, by the way, decreases in this range as n increases. So, let us try the trick

$$U\frac{dV}{d\omega} - V\frac{dU}{d\omega} = K_1\left[U^2 + V^2\right] + K_2\,\omega^{2n} \tag{4}$$

where K_1 and K_2 are real constants, with $K_1 > 0$. By a suitable definition of a non-dimensional variable $x = \omega/K_1$, we may rewrite (4) in the form

$$UV' - VU' = U^2 + V^2 + Kx^{2n} \tag{5}$$

where the symbol "prime" represents now the derivative with respect to x and K is another real constant. Since K is constant, we have

$$\frac{d}{dx}\left\{\frac{1}{x^{2n}}\left[UV' - VU' - U^2 - V^2\right]\right\} = \frac{dK}{dx} = 0 \tag{6}$$

Performing the derivative, we get the equation

$$U\left[xV'' - 2nV' - 2xU' + 2nU\right] + V\left[-xU'' + 2nU' - 2xV' + 2nV\right] = 0 \tag{7}$$

that can be written in the form

$$\Re\left[\mathcal{P}^\star\, E\right] = 0 \tag{8}$$

where E is the complex number defined by

$$\Re[E] = xV'' - 2nV' - 2xU' + 2nU, \;\; \Im[E] = -xU'' + 2nU' - 2xV' + 2nV, \text{ hence}$$
$$E = jx(-U'' - jV'') - 2jx(V' - jU') - 2n(V' - jU') + 2n(U + jV) \tag{9}$$

Now, using the variable $s = jx$, the derivatives of $\mathcal{P}$ with respect to s are

$$\frac{d\mathcal{P}}{ds} = -jU' + V', \;\; \frac{d^2\mathcal{P}}{ds^2} = -U'' - jV'' \tag{10}$$

It then follows that

$$E = s\frac{d^2\mathcal{P}}{ds^2} - 2(s+n)\frac{d\mathcal{P}}{ds} + 2n\mathcal{P} \tag{11}$$

Given (8), it seems natural, discarding other exotic possibilities, to search for polynomials satisfying $E \equiv 0$. This leads us to the equation

$$s\frac{d^2\mathcal{P}}{ds^2} - 2(s+n)\frac{d\mathcal{P}}{ds} + 2n\mathcal{P} = 0 \tag{12}$$

which is exactly that of Bessel polynomials $\theta_n(s)$, whose mathematical properties are extensively described in Ref [**3**]. This proves the optimality of Bessel polynomials as regards the desired flatness property of group delay.

To go further and noticing that $\tau(\omega)$ does not depend on the normalization of $\mathcal{P}$, let us take[2]

$$\mathcal{P}(s) = \theta_n(s) = \frac{1}{2^n}\sum_{k=0}^{n}\frac{(2n-k)!}{n!(n-k)!}(2s)^k \tag{13}$$

Then, it is seen that the highest degree term s^n in this sum has coefficient 1. This means that the term which is lacking in $UV' - VU'$ to match $U^2 + V^2$ is exactly x^{2n}. Consequently, we must set $K = -1$ in (5). We then arrive to the remarkable formula

$$\frac{d\psi}{dx}(x) = 1 - \frac{x^{2n}}{|\theta_n(jx)|^2} \tag{14}$$

valid for Bessel polynomials only, and potentially useful for practical purposes as already pointed out by I.M. Filanovsky in Ref [**2**] (formula (28)). That formula provides a simple way to check the said flatness property by means of its last term, $\eta = \dfrac{x^{2n}}{|\theta_n(jx)|^2}$, computed for some characteristic value of x. The latter is taken as 1 in table I, where it is shown how $\eta(1)$ hugely decreases as n is increasing. Finally, it is easy to deduce from (14) that in the other extreme limit $x \gg 1$, the group delay is decreasing as x increases, according to

$$\frac{d\psi}{dx}(x) \simeq \frac{n(n+1)}{2x^2} \tag{15}$$

We end here, further studies, of $\eta(x)$ in particular, being beyond the scope of this note.

n	1	2	3	4	5	6
$\eta(1) = 1/\lvert\theta_n(j)\rvert^2$	0.5	$7.7\,10^{-2}$	$3.6\,10^{-3}$	$7.8\,10^{-5}$	10^{-6}	$8.4\,10^{-9}$

Table I : $\eta(1)$ as a function of n for $n \leq 6$

[2]Of course, one may finally define the transfer function as $H(s) = \theta_n(0)/\theta_n(s)$, in order to have $H(0) = 1$.

REFERENCES

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